

Lifting KK-elements, asymptotical unitary equivalence and classification of simple C*-algebras

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Abstract

Let A and C be two unital simple C*-algebras with tracial rank zero. Suppose that C is amenable and satisfies the Universal Coefficient Theorem. Denote by $KK_e(C, A)^{++}$ the set of those κ for which $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ and $\kappa([1_C]) = [1_A]$. Suppose that $\kappa \in KK_e(C, A)^{++}$. We show that there is a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] = \kappa$. Suppose that C is a unital AH-algebra and $\lambda : T(A) \rightarrow T_f(C)$ is a continuous affine map for which $\tau(\kappa([p])) = \lambda(\tau)(p)$ for all projections p in all matrix algebras of C and any $\tau \in T(A)$, where $T(A)$ is the simplex of tracial states of A and $T_f(C)$ is the convex set of faithful tracial states of C . We prove that there is a unital monomorphism $\phi : C \rightarrow A$ such that ϕ induces both κ and λ .

Suppose that $h : C \rightarrow A$ is a unital monomorphism and $\gamma \in \text{Hom}(K_1(C), \text{Aff}(A))$. We show that there exists a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] = [h]$ in $KK(C, A)$, $\tau \circ \phi = \tau \circ h$ for all tracial states τ and the associated rotation map can be given by γ . Denote by $KKT(C, A)^{++}$ the set of compatible pairs (κ, λ) , where $\kappa \in KL_e(C, A)^{++}$ and λ is a continuous affine map from $T(A)$ to $T_f(C)$. Together with a result of asymptotic unitary equivalence in [15], this provides a bijection from the asymptotic unitary equivalence classes of unital monomorphisms from C to A to $(KKT(C, A)^{++}, \text{Hom}(K_1(C), \text{Aff}(T(A)))/\mathcal{R}_0)$, where \mathcal{R}_0 is a subgroup related to vanishing rotation maps.

As an application, combining with a result of W. Winter ([24]), we show that two unital amenable simple \mathcal{Z} -stable C*-algebras are isomorphic if they have the same Elliott invariant and the tensor products of these C*-algebras with any UHF-algebras have tracial rank zero. In particular, if A and B are two unital separable simple \mathcal{Z} -stable C*-algebras which are inductive limits of C*-algebras of type I with unique tracial states, then they are isomorphic if and only if they have isomorphic Elliott invariant.

1 Introduction

Let A and B be two unital separable amenable simple C*-algebras satisfying the Universal Coefficient Theorem. It has been shown (see [12] and [17], also [2] and [3]) that, if in addition, A and B have tracial rank one or zero, then A and B are isomorphic if their Elliott invariant are isomorphic. There are interesting simple amenable C*-algebras with stable rank one which do not have tracial rank zero or one and some classification theorem have been established too (see for example [7], [6], [20] and [5]). One of the interesting classes of simple amenable C*-algebras which satisfy the UCT are those simple ASH-algebras (approximate sub-homogeneous C*-algebras), or even more general, those simple C*-algebras which are inductive limits of type I C*-algebras. In the case of

unital \mathcal{Z} -stable simple ASH-algebras, if in addition, their projections separate traces, a classification theorem can be given (see [24] and [16]). More precisely, let A and B be two unital simple \mathcal{Z} -stable ASH-algebras whose projections separate the traces. Then $A \cong B$ if and only if $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ is isomorphic to $(K_0(B), K_0(B)_+, [1_B], K_1(B))$ provided that $K_i(A)$ are finitely generated, or $K_i(A)$ contains its torsion part as a direct summand. It should be noted that there are known examples that ASH-algebras whose projections separate the traces but have real rank other than zero. In particular, A and B may not have any non-trivial projections as long as each one has a unique tracial state (for instance, the Jiang-Su algebra \mathcal{Z}). It is clearly important to remove the above mentioned restriction on K -theory. The original purpose of this paper was to remove these restrictions.

One of the technical tools used in the proof of [24] and [16] is a theorem which determines when two unital monomorphisms from a unital AH-algebra C to a unital simple C^* -algebra A are asymptotically unitarily equivalent. In the case that A has tracial rank zero, it was shown in [15] that two such monomorphisms are asymptotically unitarily equivalent if they induce the same element in $KK(C, A)$ and the same affine map on the tracial state spaces, and the rotation map associated with these two monomorphisms vanishes. It is equally important to determine the range of asymptotical unitary equivalence classes of those monomorphisms. In fact, the above mentioned restriction can be removed if the range can be determined.

The first question we need to answer is the following: Let A and C be two unital simple amenable C^* -algebras with tracial rank zero and let C satisfy the UCT. Suppose that $\kappa \in KK_e(C, A)^{++}$ (i.e., those $\kappa \in KK(C, A)$ such that $\kappa(K_0(C)_+ \setminus \{0\}) \subseteq K_0(A)_+ \setminus \{0\}$ with $\kappa([1_C]) = [1_A]$). Is there a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] = \kappa$? It is known (see [9]) that if $K_i(C)$ is finitely generated then the answer is affirmative. It was proved in [10] that there exists a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] - \kappa = 0$ in $KL(C, A)$. The problem remained open whether one can choose ϕ so that $[\phi] = \kappa$ in $KK(C, A)$. It also has been known that there are several significant consequences if such ϕ can be found.

The second question is the following: Let $\phi : C \rightarrow A$ be a unital monomorphism and $\gamma \in \text{Hom}(K_1(C), \text{Aff}(T(A)))$. Can we find a unital monomorphism $\psi : C \rightarrow B$ with $[\psi] = [\phi]$ in $KK(C, A)$ and the associated rotation map is γ ?

In this paper, we will give affirmative answers to both questions. Among other consequences, we give the following: Let A and B be two unital simple C^* -algebras which are inductive limits of type I C^* -algebras with unique tracial states. Suppose that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Then A and B are \mathcal{Z} -stably isomorphic (see 5.6 below).

For the first question, as mentioned above, an affirmative answer was known for C with finitely generated K -theory. Passing to inductive limits, the first author showed in [10] that any strictly positive element in $KL(C, A)$ can be lift for any unital AH-algebra C . However, since the KK -functor is not continuous with respect to inductive limits, it remained somewhat mysterious how to move from $KL(C, A)$ to $KK(C, A)$ until a clue was given by a paper of Kishimoto and Kumjian ([8]), where they studied simple $A\mathbb{T}$ -algebras with real rank zero. The important advantage of simple $A\mathbb{T}$ -algebras is that their K -theory are torsion free. In this case a hidden Bott-like map was revealed. Kishimoto and Kumjian navigated this hurdle using the so-called Basic Homotopy Lemma in simple C^* -algebras of real rank zero. We will take the idea of Kishimoto and Kumjian to the case that

the domain algebras are no longer assumed the same as the targets. More generally, we will not assume that C has real rank zero, nor will we assume it is simple. Furthermore, we will allow C to have torsion in its K -theory. Therefore, for the general case, the Bott-like maps involve the K -theory with coefficient and demand a much more general Basic Homotopy Lemma. For this we apply the recent established results in [14] and [15]. It is also interesting to note that the second problem is closely related to the first one and its proof are also closely related.

We will consider the case that C is a general unital AH-algebra (which may have arbitrary stable rank and other properties even in the case it is simple (see, for example, [22], [23] and [4])). Denote by $T_f(C)$ the convex set of all faithful tracial states of C . Let $\kappa \in KK_e(C, A)^{++}$ and let $\lambda : T(A) \rightarrow T_f(C)$ be a continuous affine map. We say that λ is compatible with κ if $\lambda(\tau)(p) = \tau(\kappa([p]))$ for all projections p in matrix algebras of C and for all $\tau \in T(A)$. We actually show that for any such compatible pair (κ, λ) , there exists a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] = \kappa$ and $\phi_T = \lambda$, where $\phi_T : T(A) \rightarrow T_f(C)$ is the induced continuous affine map induced by ϕ . It worth to point out that the information λ is essential since there are examples of compact metric spaces X , unital simple AF-algebras and $\kappa \in K_e(C, A)^{++}$ for which there is no unital monomorphism h such that $[h] = \kappa$ (see [18]). Furthermore, given a pair $([\phi], \phi_T)$ and $\gamma \in \text{Hom}(K_1(C), \text{Aff}(T(A)))$, there is a unital monomorphism $\psi : C \rightarrow A$ such that $([\psi], \psi_T) = ([\phi], \phi_T)$ and a rotation map from $K_1(C)$ to $\text{Aff}(T(A))$ associated with ϕ and ψ is exactly γ .

The paper is organized as follows: Preliminaries and notation are given in Section 2. In Section 3, it is shown that any element $\kappa \in KK_e(C, A)^{++}$ for which is compatible with a continuous affine map $\lambda : T(A) \rightarrow T_f(C)$ can be represented by a monomorphism α if C is a unital AH-algebra and A is a unital simple C^* -algebra with tracial rank zero. It is also shown that if $A = C$, then α can be chosen as an automorphism. Then, in Section 4, we prove that, for any monomorphism $\iota : C \rightarrow A$, one can realize any homomorphism ψ from $K_1(C)$ to $\text{Aff}(T(A))$ as a rotation map without changing the KK -class of ι , that is, there is a monomorphism $\alpha : C \rightarrow A$ such that $[\iota] = [\alpha]$ in $KK(C, A)$ and $\tilde{\eta}_{\iota, \alpha} = \psi$. Moreover, we also give a description of the asymptotical unitary equivalence class of the maps inducing the same KK -element. In Section 5, we give an application of the results in the previous sections to the classification program. Combined with the work [24] of W. Winter and that of [15], it is shown that certain \mathcal{Z} -stable C^* -algebras can be classified by their K -theory information.

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2 Preliminaries and Notation

2.1. Let A be a unital stably finite C^* -algebra. Denote by $T(A)$ the simplex of tracial states of A and denote by $\text{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. Suppose that $\tau \in T(A)$ is a tracial state. We will also use τ for the trace $\tau \otimes \text{Tr}$ on $M_k(A) = A \otimes M_k(\mathbb{C})$ (for every integer $k \geq 1$), where Tr is the standard trace on $M_k(\mathbb{C})$. A trace τ is faithful if $\tau(a) > 0$ for any $a \in A_+ \setminus \{0\}$. Denote by $T_f(A)$ the convex subset of $T(A)$ which consists of all faithful tracial states.

Denote by $M_\infty(A)$ the set $\bigcup_{k=1}^{\infty} M_k(A)$, where $M_k(A)$ is regarded as a C^* -subalgebra of $M_{k+1}(A)$ by the

embedding

$$M_k(A) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{k+1}(A)$$

Define the positive homomorphism $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$ by $\rho_A([p])(\tau) = \tau(p)$ for any projection p in $M_k(A)$. We also denote by $S(A)$ the suspension of A , $U(A)$ the unitary group of A , and $U_0(A)$ the connected component of $U(A)$ containing the identity.

Suppose that C is another unital C^* -algebra and $\phi : C \rightarrow A$ is a unital $*$ -homomorphism. Denote by $\phi_T : T(A) \rightarrow T(C)$ the continuous affine map induced by ϕ , i.e., $\phi_T(\tau)(c) = \tau \circ \phi(c)$ for all $c \in C$ and $\tau \in T(A)$.

Definition 2.2. Let A be a unital C^* -algebra and let $B \subseteq A$ be a unital C^* -subalgebra. For any $u \in U(A)$, the $*$ -homomorphism $\text{Ad}(u)$ is defined by

$$\text{Ad}(u) : B \ni b \mapsto u^*bu \in A.$$

Denote by $\overline{\text{Inn}}(B, A)$ the closure of $\{\text{Ad}(u); u \in U(A)\}$ in $\text{Hom}(B, A)$ with the pointwise convergence topology. Note that $\overline{\text{Inn}}(A) \subseteq \overline{\text{Inn}}(B, A)$.

Definition 2.3. Let A and B be two unital C^* -algebras, and let ψ and ϕ be two unital $*$ -monomorphisms from B to A . Then the mapping torus $M_{\psi, \phi}$ is the C^* -algebra defined by

$$M_{\psi, \phi} := \{f \in C([0, 1], A); f(0) = \psi(b) \text{ and } f(1) = \phi(b) \text{ for some } b \in B\}.$$

If $B \subseteq A$ is a unital C^* -subalgebra with ι the inclusion map, then, for any unital $*$ -monomorphism α from B to A , we also denote $M_{\iota, \alpha}$ by M_α .

For any $\psi, \phi \in \text{Hom}(B, A)$, denoting by π_0 the evaluation of $M_{\psi, \phi}$ at 0, we have the short exact sequence

$$0 \longrightarrow S(A) \xrightarrow{\iota} M_{\psi, \phi} \xrightarrow{\pi_0} B \longrightarrow 0,$$

and hence the six-term exact sequence

$$\begin{array}{ccccc} K_0(S(A)) & \xrightarrow{\iota_0} & K_0(M_{\psi, \phi}) & \xrightarrow{[\pi_0]_0} & K_0(B) \\ [\psi]_1 - [\phi]_1 \uparrow & & & & \downarrow [\psi]_0 - [\phi]_0 \\ K_1(B) & \xleftarrow{[\pi_0]_1} & K_1(M_{\psi, \phi}) & \xleftarrow{\iota_1} & K_1(S(A)). \end{array}$$

If $[\psi]_* = [\phi]_*$, (in particular, if $B \subseteq A$ and $\alpha \in \overline{\text{Inn}}(B, A)$), then the six-term exact sequence above breaks down to the following two extensions:

$$\eta_0(M_{\psi, \phi}) : 0 \longrightarrow K_1(A) \longrightarrow K_0(M_{\psi, \phi}) \longrightarrow K_0(B) \longrightarrow 0,$$

and

$$\eta_1(M_{\psi, \phi}) : 0 \longrightarrow K_0(A) \longrightarrow K_1(M_{\psi, \phi}) \longrightarrow K_1(B) \longrightarrow 0.$$

Moreover, if $[\psi] = [\phi]$ in $KL(B, A)$, the two extensions above are pure.

2.4. Suppose that, in addition,

$$\tau \circ \phi = \tau \circ \psi \text{ for all } \tau \in T(A). \tag{2.1}$$

For any piecewise smooth path $u(t) \in M_{\psi,\phi}$, the integral

$$R_{\phi,\psi}(u(t))(\tau) = \frac{1}{2\pi i} \int_0^1 \tau(\dot{u}(t)u^*(t))dt$$

defines an affine function on $T(A)$, and it depends only on the homotopy class of $u(t)$. Therefore, it induces a map, denoted by $R_{\psi,\phi}$, from $K_1(M_\alpha)$ to $\text{Aff}(T(A))$, and we call it the rotation map. The map $R_{\psi,\phi}$ is in fact a homomorphism.

2.5. If p and q be two mutually orthogonal projections in $M_l(A)$ for some integer $l \geq 1$, define a unitary $u \in U(\widetilde{M_l(S(A))})$ by

$$u(t) = (e^{2\pi i t}p + (1-p))(e^{-2\pi i t}q + (1-q)) \text{ for } t \in [0, 1].$$

One computes that

$$\int_0^1 \tau\left(\frac{du(t)}{dt}u(t)^*\right)dt = \tau(p) - \tau(q) \text{ for all } t \in [0, 1].$$

Note that if $v(t) \in \widetilde{M_l(SA)}$ is another piecewise smooth unitary which is homotopic to $u(t)$, then, as mentioned above,

$$\int_0^1 \tau\left(\frac{dv(t)}{dt}\right)dt = \tau(p) - \tau(q).$$

It follows that, for any two projections p and q in $M_l(A)$,

$$R_{\phi,\psi}(\iota_1([p] - [q]))(\tau) = \tau(p) - \tau(q) \text{ for all } \tau \in T(A).$$

In other words,

$$R_{\phi,\psi}(\iota_1([p] - [q])) = \rho_A([p] - [q]).$$

Thus one has, exactly as in 2.2 of [8], the following:

Lemma 2.6. *When (2.1) holds, the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\iota_1} & K_1(M_{\phi,\psi}) \\ \rho_A \searrow & & \swarrow R_{\phi,\psi} \\ & \text{Aff}(T(A)) & \end{array}$$

Definition 2.7. Let A and C be two unital C^* -algebras and let $\phi, \psi : C \rightarrow A$ be two unital homomorphisms. We say that ϕ and ψ are asymptotically unitarily equivalent if there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\}$ such that

$$\lim_{t \rightarrow \infty} \text{Ad}(u(t)) \circ \phi(c) = \psi(c) \text{ for all } c \in C.$$

Definition 2.8. Let A be a unital C^* -algebra and let C be a separable C^* -algebra which satisfies the Universal Coefficient Theorem. By [1] of Dădărlat and Loring,

$$KL(C, A) = \text{Hom}_\Lambda(\underline{K}(C), \underline{K}(A)), \quad (2.2)$$

where, for any C^* -algebra B ,

$$\underline{K}(B) = (K_0(B) \oplus K_1(B)) \oplus \left(\bigoplus_{n=2}^{\infty} (K_0(B, \mathbb{Z}/n\mathbb{Z}) \oplus K_1(B, \mathbb{Z}/n\mathbb{Z})) \right).$$

We will identify the two objects in (2.2). Note that one may view $KL(C, A)$ as a quotient of $KK(C, A)$.

Denote by $KL(C, A)^{++}$ the set of those $\bar{\kappa} \in \text{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ such that

$$\bar{\kappa}(K_0(C)_+ \setminus \{0\}) \subseteq K_0^+(A) \setminus \{0\}.$$

Denote by $KL_e(C, A)^{++}$ the set of those elements $\bar{\kappa} \in KL(C, A)^{++}$ such that $\bar{\kappa}([1_C]) = [1_A]$.

Suppose that both A and C are unital and $T(C) \neq \emptyset$ and $T(A) \neq \emptyset$. Let $\lambda : T(A) \rightarrow T(C)$ be a continuous affine map. We say λ is compatible with $\bar{\kappa}$ if for any projection $p \in M_\infty(C)$, one has that $\lambda(\tau)(p) = \tau(\bar{\kappa}([p]))$ for all $\tau \in T(A)$.

Denote by $KL T(C, A)^{++}$ the set of those pairs $(\bar{\kappa}, \lambda)$, where $\bar{\kappa} \in KL_e(C, A)^{++}$ and $\lambda : T(A) \rightarrow T_\mathbf{f}(C)$ is a continuous affine map which is compatible with $\bar{\kappa}$.

Definition 2.9. Denote by $KK(C, A)^{++}$ the set of those elements $\kappa \in KK(C, A)$ such that its image $\bar{\kappa}$ is in $KL(C, A)^{++}$. Denote by $KK_e(C, A)^{++}$ the set of those $\kappa \in KK(C, A)^{++}$ for which $\bar{\kappa} \in KL_e(C, A)^{++}$.

Denote by $KK T(C, A)^{++}$ the set of pairs (κ, λ) such that $(\bar{\kappa}, \lambda) \in KL T(C, A)^{++}$.

2.10. Let A and B be two unital C^* -algebras. Let $h : A \rightarrow B$ be a homomorphism and $v \in U(B)$ such that

$$[h(g), v] = 0 \quad \text{for any } g \in A.$$

We then have a homomorphism $\bar{h} : A \otimes C(\mathbb{T}) \rightarrow B$ by $f \otimes g \mapsto h(f)g(v)$ for any $f \in A$ and $g \in C(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \rightarrow K_1(A \otimes C(\mathbb{T})),$$

and

$$\beta^{(1)} : K_1(A) \rightarrow K_0(A \otimes C(\mathbb{T})).$$

The second one is the usual Bott map. Note, in this way, one writes

$$K_i(A \otimes C(\mathbb{T})) = K_i(A) \oplus \beta^{(i-1)}(K_{i-1}(A)).$$

We use $\widehat{\beta^{(i)}} : K_i(A \otimes C(\mathbb{T})) \rightarrow \beta^{(i-1)}(K_i(A))$ for the projection.

For each integer $k \geq 2$, one also have the following injective homomorphisms:

$$\beta_k^{(i)} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \rightarrow K_{i-1}(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1.$$

Thus, we write

$$K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) = K_i(A, \mathbb{Z}/k\mathbb{Z}) \oplus \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z}).$$

Denote by $\widehat{\beta_k^{(i)}} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \rightarrow \beta^{(i-1)}(K_i(A), \mathbb{Z}/k\mathbb{Z})$ similar to that of $\widehat{\beta^{(i)}}$. If $x \in \underline{K}(A)$, we use $\beta(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\beta : \underline{K}(A) \rightarrow \underline{K}(A \otimes C(\mathbb{T}))$ as well as $\widehat{\beta} : \underline{K}(A \otimes C(\mathbb{T})) \rightarrow \beta(\underline{K})$. Therefore, we may write $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \beta(\underline{K}(A))$.

On the other hand, \bar{h} induces homomorphisms

$$\bar{h}_{*,i,k} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \rightarrow K_i(B, \mathbb{Z}/k\mathbb{Z}),$$

$k = 0, 2, \dots$, and $i = 0, 1$.

We use $\text{Bott}(h, v)$ for all homomorphisms $\bar{h}_{*i,k} \circ \beta_k^{(i)}$, and we use $\text{bott}_1(h, v)$ for the homomorphism $\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \rightarrow K_0(B)$, and $\text{bott}_0(h, v)$ for the homomorphism $\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \rightarrow K_1(B)$. We also use $\text{bott}(u, v)$ for the Bott element when $[u, v] = 0$.

2.11. Given a finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$\text{Bott}(h, v)|_{\mathcal{P}}$$

is well defined if

$$\|[h(a), v]\| < \delta_0$$

for all $a \in \mathcal{F}$. See 2.11 of [15] and 2.10 of [14] for more details.

Definition 2.12. A unital simple C*-algebra A has tracial rank zero, denoted by $\text{TR}(A) = 0$, if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, and nonzero $a \in A^+$, there are nonzero projection $p \in A$ and finite dimensional C*-subalgebra F with $1_F = p$, such that

1. $\|[x, p]\| \leq \varepsilon$ for any $x \in \mathcal{F}$,
2. for any $x \in \mathcal{F}$, there is $x' \in F$ such that $\|pxp - x'\| \leq \varepsilon$, and
3. $1 - p$ is Murray-von Neumann equivalent to a projection in \overline{aAa} .

2.13. Finally, we will write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$.

3 KK-lifting

Let A be a unital C*-algebra. Fix $0 < \delta^p \leq \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are the constant of δ of Lemma 9.6 and Lemma 9.7 of [15] respectively. Note that δ^p is universal and $\delta^p < 1/4$ (therefore, the Bott element $\text{bott}(u, v)$ is well-defined for any unitaries u and v with $\|[u, v]\| < \delta^p$). For some integer $l \geq 1$, let z be a unitary in $M_l(A)$, and let $U(t)$ be a path of uniaries $U(t) \in C([0, 1], U(M_l(A)))$ with $U(0) = 1$ and $\|[U(1), z]\| \leq \delta^p$, we have that

$$U(1)zU^*(1)z^* = \exp(i\omega)$$

where $\omega = (1/i)(\log(U(1)zU^*(1)z^*)) \in A_{s.a.}$. Define the element $\mathbf{p}(U, z)$ as follows:

$$\mathbf{p}(U, z)(t) = \begin{cases} U((8/7)t)zU^*((8/7)t)z^* & \text{for all } t \in [0, 7/8]; \\ \exp(i8(1-t)\omega) & \text{for all } t \in [7/8, 1] \end{cases}.$$

Then, $\mathbf{p}(U, z)(t)$ defines a loop of unitaries in A , and this gives a well-defined element in $K_1(S(A)) \cong K_0(A)$.

Remark 3.1. For a path of uniaries $U(t) \in C([0, 1], U(M_l(A)))$ for some integer $l \geq 1$. Denote by

$$S(t) := \begin{pmatrix} U(t) & \\ & U^*(t) \end{pmatrix} \quad \text{and} \quad Z := \begin{pmatrix} z & \\ & z \end{pmatrix}.$$

If $U(0) = 1$ and $\|[U(1), z]\| \leq \delta^p$, then $S(0) = 1$ and $\|[S(1), Z]\| \leq \delta^p$, and we can consider the K -element $[\mathbf{p}(S, Z)]$ and we have

$$[\mathbf{p}(S, Z)] = [\mathbf{p}(U, z)] + [\mathbf{p}(U^*, z)].$$

Denote by

$$R(t, s) := \begin{pmatrix} U(t) & \\ & 1 \end{pmatrix} R(s) \begin{pmatrix} 1 & \\ & U^*(t) \end{pmatrix} R^*(s)$$

where

$$R(s) = \begin{pmatrix} \cos(\frac{\pi s}{2}) & \sin(\frac{\pi s}{2}) \\ -\sin(\frac{\pi s}{2}) & \cos(\frac{\pi s}{2}) \end{pmatrix}.$$

Then one has that $R(t, 0) = S(t)$, $R(t, 1) = 1$, and for any $s \in [0, 1]$, $R(0, s) = 1$ and $\|[R(1, s), Z]\| \leq 2\delta^p$. Therefore, the path $\mathbf{p}(S, Z)(t)$ is homotopic to the identity in $M_2(\widetilde{S(A)})$ by a small perturbation of

$$W(t, s) := R(t, s)ZR^*(t, s)Z^*,$$

and hence

$$[\mathbf{p}(U, z)] + [\mathbf{p}(U^*, z)] = [\mathbf{p}(S, Z)] = 0 \text{ in } K_1(S(A)).$$

Lemma 3.2. *Let $U(t)$ and $V(t)$ be two continuous and piecewise smooth paths of unitaries. Let z_1 and z_2 be unitaries in A with $\|[U(1), z_1]\| < \delta^p$ and $\|[V(1), z_2]\| < \delta^p$. If $[\mathbf{p}(U, z_1)] = [\mathbf{p}(V, z_2)]$, then one has that $\text{bott}(U(1), z_1) = \text{bott}(V(1), z_2)$.*

Proof. Consider the path of unitaries

$$S(t) = \text{diag}(U(t), V^*(t))$$

and the unitary

$$Z := \text{diag}(z_1, z_2)$$

in $M_2(A)$. By Remark 3.1, one has

$$[\mathbf{p}(V^*, z_2)] + [\mathbf{p}(V, z_2)] = 0,$$

and hence

$$[\mathbf{p}(S^*, Z)] = [\mathbf{p}(U^*, z_1)] + [\mathbf{p}(V, z_2)] = 0.$$

Therefore, $\mathbf{p}(S^*, Z)$ is homotopic to the identity in a matrix algebra of $\widetilde{S(A)}$. Hence, passing to the matrix algebra if necessary, there is a continuous path of unitaries $W(t, s)$ such that $W(t, 0) = 1 = W(t, 1)$ for any $t \in [0, 1]$ and $W(1, t) = \mathbf{p}(S^*, Z)(t)$ for any $t \in [0, 1]$, and $W(t, 1) = 1$.

Note that $\mathbf{p}(S^*, Z)(t) = S^*(\frac{8}{7}t)ZS(\frac{8}{7}t)Z^*$ for $t \in [0, 7/8]$ and

$$\|\mathbf{p}(S^*, Z)(7/8) - 1\| < \delta^p.$$

Thus, by Lemma 9.6 of [15],

$$0 = \text{bott}(S(1), Z) = \text{bott}(U(1), z_1) - \text{bott}(V(1), z_2),$$

as desired. □

Lemma 3.3. *Let $U(t)$ and $V(t)$ be two continuous and piecewise smooth paths of unitaries. Let z_1 and z_2 be unitaries in A with $\|[U(t), z_1]\| < \delta^p$ and $\|[V(t), z_2]\| < \delta^p$. If $\text{bott}(U(1), z_1) = \text{bott}(V(1), z_2)$, then one has that $[\mathbf{p}(U, z_1)] = [\mathbf{p}(V, z_2)]$.*

Proof. Consider the path of unitaries

$$W(t) := \text{diag}(U(t), V^*(t))$$

and unitary

$$Z := \text{diag}(z_1, z_2)$$

in $M_2(A)$. Then, one has that $\text{bott}(W(1), Z) = \text{bott}(U(1), z_1) - \text{bott}(V(1), z_2) = 0$. By Lemma 9.7 of [15], one has

$$[\mathbf{p}(W^*, Z)] = 0,$$

and hence

$$[\mathbf{p}(U^*, z_1)] + [\mathbf{p}(V, z_2)] = 0. \quad (3.1)$$

On the other hand, by Remark 3.1,

$$[\mathbf{p}(V^*, z_2)] + [\mathbf{p}(V, z_2)] = 0.$$

Together with (3.1), one has

$$[\mathbf{p}(U, z_1)] = [\mathbf{p}(V, z_2)],$$

as desired. \square

3.4. Denote by $C(A)$ the subset of $K_1(S(A))$ consisting of $[\mathbf{p}(U, z)(t)]$ for a path of unitaries $U(t) \in C([0, 1], M_l(A))$ and a unitary $z \in M_l(A)$ for some $l \geq 1$ with $\|[U(1), z]\| < \delta^{\mathbf{p}}$. It is easy to verify that $C(A)$ is a subgroup. It follows from Lemma 3.2 and Lemma 3.3 that there is an injective map $\Lambda : C(A) \rightarrow K_0(A)$ defined by

$$\Lambda : [\mathbf{p}(U, z)] \mapsto \text{bott}(U(1), z).$$

Moreover, the map Λ is a homomorphism.

Lemma 3.5. *Consider a path of unitaries $U(t) \in C([0, 1], M_\infty(A))$ with $U(0) = 1$ and $\|[z, U(1)]\| < \delta^{\mathbf{p}}$. Then we have*

$$\tau([\mathbf{p}(U, z)]) = \tau(\text{bott}_1(U(1), z))$$

for any $\tau \in T(A)$ if one considers $[\mathbf{p}(U, z)]$ as an element in $K_0(A) \cong K_1(S(A))$. In other words, $\tau(\Lambda(h)) = \tau(h)$ for any $h \in C(A)$.

Proof. Denote by $\omega = U(1)zU^*(1)z^*$, and note that $[\mathbf{p}(U, z)] \in K_1(S(A)) = K_0(A)$. Denote by

$$r(t) = \exp(\log(\omega)(1 - t)).$$

We then have that (see 2.5), for any $\tau \in T(A)$,

$$\begin{aligned}
\tau(\mathbf{p}(U, z)) &= \frac{1}{2\pi i} \int_0^{7/8} \tau((U((8/7)t)zU^*((8/7)t)z^*)'(zU((8/7)t)z^*U^*((8/7)t))dt \\
&\quad + \frac{1}{2\pi i} \int_{7/8}^1 \tau(\dot{r}(8(t-7/8))r^*(8(t-7/8)))dt \\
&= \frac{1}{2\pi i} \int_0^{7/8} \tau((U((8/7)t)zU^*((8/7)t))'(U((8/7)t)z^*U^*((8/7)t))dt \\
&\quad + \frac{1}{2\pi i} \int_{7/8}^1 \tau(\dot{r}(8(t-7/8))r^*(8(t-7/8)))dt \\
&= \frac{1}{2\pi i} \int_{7/8}^1 \tau(\dot{r}(8(t-7/8))r^*(8(t-7/8)))dt \quad (\text{by Lemma 4.2 of [15]}) \\
&= -\frac{1}{2\pi i} \tau(\log(w)) \\
&= \tau(\text{bott}(U(1), z)) \quad (\text{by Theorem 3.6 of [15]}),
\end{aligned}$$

as desired. \square

Definition 3.6. Let A be a unital C^* -algebra with $T(A) \neq \emptyset$. We say that A has Property (B1) if the following holds: For any unitary $z \in U(M_k(A))$ (for some integer $k \geq 1$) with $\text{sp}(z) = \mathbb{T}$, there is a non-decreasing function $1/4 > \delta_z(t) > 0$ on $[0, 1]$ with $\delta_z(0) = 0$ such that for any $x \in K_0(A)$ with $|\tau(x)| \leq \delta_z(\varepsilon)$ for all $\tau \in T(A)$, there exists a unitary $u \in M_k(A)$ such that

$$\|[u, z]\| < \min\{\varepsilon, \frac{1}{4}\} \quad \text{and} \quad \text{bott}_1(u, z) = x. \quad (3.2)$$

Let C be a unital separable C^* -algebra. Let $1/4 > \Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) > 0$ be a function defined on $t \in [0, 1]$, the family of all finite subsets $\mathcal{F} \subset C$, the family of all finite subsets $\mathcal{P}_0 \subset K_0(C)$, and family of all finite subsets $\mathcal{P}_1 \subset K_1(C)$, and the set of all unital monomorphisms $h : C \rightarrow A$. We say that A has Property (B2) associated with C and Δ_c if the following holds:

For any unital monomorphism $h : C \rightarrow A$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C$, any finite subset $\mathcal{P}_0 \subset K_0(C)$, and any finite subset $\mathcal{P}_1 \subset K_1(C)$, there are finitely generated subgroups $G_0 \subset K_0(C)$ and $G_1 \subset K_1(C)$ with \mathcal{G}_0 and \mathcal{G}_1 the sets of generators respectively and $\mathcal{P}_0 \subset G_0$ and $\mathcal{P}_1 \subset G_1$ satisfying the following: for any homomorphisms $b_0 : G_0 \rightarrow K_1(A)$ and $b_1 : G_1 \rightarrow K_0(A)$ such that

$$|\tau \circ b_1(g)| < \Delta_c(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) \quad (3.3)$$

for any $g \in \mathcal{G}_1$ and any $\tau \in T(A)$, there exists a unitary $u \in U(A)$ such that

$$\text{bott}_0(h, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \quad \text{bott}_1(h, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad \text{and} \quad (3.4)$$

$$\|[h(c), u]\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \quad (3.5)$$

Remark 3.7. Note that in the definition of Property (B2), b_0 and b_1 are defined on G_0 and G_1 , respectively, not on the subgroups generated by \mathcal{P}_0 and \mathcal{P}_1 . One should also note that if A has Property (B2) associated with any C^* -subalgebra $C(\mathbb{T})$, then A also has Property (B1).

Remark 3.8. Let A be a C^* -algebra with Property (B2) associated with C and $\Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h)$. Then the function Δ'_c and the subgroups G_0 and G_1 can be chosen so that they only depend on the unitary conjugate class of the unital embedding h .

Indeed, pick one representative h_λ for each conjugate class of the unital embedding of C to A , and define the function

$$\Delta'_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) = \Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h_\lambda)$$

where h_λ is in the unitary conjugate class of h . It is clear that Δ'_c only depends on the conjugate class of h . Let us show that A has Property (B2) associated with C and Δ'_c and G_0 and G_1 can be chosen so that they only depend on h_λ .

Fix a unital monomorphism $h : C \rightarrow A$, any $\varepsilon > 0$, a finite subset $\mathcal{F} \subset C$, a finite subset $\mathcal{P}_0 \subset K_0(C)$, and a finite subset $\mathcal{P}_1 \subset K_1(C)$. There is a unitary $w \in A$ such that $\text{Ad}(w) \circ h = h_\lambda$ for some λ .

Since A has Property (B2) associated with C and Δ_c , there are finitely generated subgroups $G_0 \subset K_0(C)$ and $G_1 \subset K_1(C)$ with \mathcal{G}_0 and \mathcal{G}_1 the sets of generators respectively and $\mathcal{P}_0 \subset G_0$ and $\mathcal{P}_1 \subset G_1$ satisfying the following: for any homomorphisms $b_0 : G_0 \rightarrow K_1(A)$ and $b_1 : G_1 \rightarrow K_0(A)$ such that

$$|\tau \circ b_1(g)| < \Delta_c(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h_\lambda) \quad (3.6)$$

for any $g \in \mathcal{G}_1$ and any $\tau \in T(A)$, there exists a unitary $u \in U(A)$ such that

$$\text{bott}_0(h_\lambda, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \quad \text{bott}_1(h_\lambda, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad \text{and} \quad (3.7)$$

$$\|[h_\lambda(c), u]\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \quad (3.8)$$

Then,

$$\text{bott}_0(h, wuw^*) = \text{bott}_0(h_\lambda, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \quad \text{bott}_1(h, wuw^*) = \text{bott}_1(h_\lambda, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad (3.9)$$

$$\text{and } \|[h(c), wuw^*]\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \quad (3.10)$$

In other words, A has Property (B2) associated with Δ'_c . Therefore, we can always assume that the function Δ_c and the subgroups G_0 and G_1 only depend on the conjugate classes of embeddings of C into A .

Lemma 3.9. *Let A be a unital C^* -algebra which contains a positive element b with $\text{sp}(b) = [0, 1]$, and assume that A has Property (B1). There exists $\delta > 0$ such that for any $a \in K_0(A)$, if $|\tau(a)| \leq \delta$ for any $\tau \in T(A)$, then one has*

$$a = [\mathbf{p}(U, z)] \in K_1(S(A))$$

for some unitary $z \in M_k(A)$ (for some $k \geq 1$) and some $U(t) \in C([0, 1], U(M_k(A)))$ with $U(0) = 1$ and $\|[U(1), z]\| \leq \delta^{\mathbf{p}}$. In other words, $a \in C(A)$.

Proof. Since A has Property (B1), for any $1/4 > \varepsilon > 0$ and any unitary z in a matrix of A with $\text{sp}(z) = \mathbb{T}$, there is a non-decreasing positive function $\delta_z(t)$ such that if $a \in K_0(A)$ with $|\tau(a)| \leq \delta_z(\varepsilon)$ for any trace τ , then there is a unitary u in the matrix of A such that $\|[u, z]\| < \varepsilon$ and $\text{bott}_1(u, z) = a$.

For any ε , there is $\delta_e(\varepsilon)$ such that if $|x - y| \leq \delta_e(\varepsilon)$ with $x, y \in \{c \in \mathbb{C}; |c - 1| \leq \frac{1}{2}\}$, one has that $|\log(x) - \log(y)| \leq \varepsilon$. We also regard $\delta_e(\varepsilon)$ as a positive function of ε with $\delta_e(0) = 0$ and $\delta_e(\varepsilon) > 0$ if $\varepsilon > 0$.

For any natural number k and any $\varepsilon > 0$, define

$$\Delta_z(\varepsilon, k) := \begin{cases} \min\{\delta_z(\frac{1}{2}\Delta_z(\varepsilon, k-1)), \frac{1}{2}\delta_e(\frac{1}{2}\Delta_z(\varepsilon, k-1)), \delta^p, \varepsilon\} & \text{if } k \geq 2, \\ \min\{\delta_z(\varepsilon), \delta^p\} & \text{if } k = 1. \end{cases}$$

It is a positive function of ε and k with $\Delta_z(\varepsilon, k) > 0$ if $\varepsilon > 0$.

Fix $0 < \varepsilon < 1/12$. Then, there exists $m \in \mathbb{N}$ such that there is a partition $0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$ such that for any $W(t) = (e^{2\pi it}p + (1-p))(e^{-2\pi it}q + (1-q)) \in U_\infty(\widetilde{S(A)})$ where p and q are any projections, one has that $\|W(t_{i-1}) - W(t_i)\| \leq \varepsilon$ for each $1 \leq i \leq m$. Fix this partition.

Let $z_0 = \exp(i2\pi b)$. Since $\text{sp}(b) = [0, 1]$, z_0 is a unitary in A with $\text{sp}(z_0) = \mathbb{T}$. Denote by

$$\delta = \frac{1}{2} \min\{\Delta_{z_0}(\varepsilon, 2m), \Delta_{z_0}(\varepsilon, 2m-1), \dots, \Delta_{z_0}(\varepsilon, 1)\}.$$

Let a be an element in $K_0(A)$ with $|\tau(a)| < \delta$ for any $\tau \in T(A)$. Without loss of generality, we may assume that $a = [p] - [q]$ for some projections $p, q \in M_k(A)$ for some integer $k \geq 1$. Note that

$$|\tau(p) - \tau(q)| < \delta.$$

Put $W_0(t) = (e^{2\pi it}p + (1-p))(e^{-2\pi it}q + (1-q))$. To simplify notation, by replacing A by $M_k(A)$, without loss of generality, we may assume that $p, q \in A$ and $W_0(t) \in A$ for each $t \in [0, 1]$.

It follows from Lemma 8 of [19] that, for some large $n \geq 1$, there is a unitary $V_t \in M_{n+1}(A)$, for each $t \in [0, 1]$,

$$\|V_t^* z V_t - W(t)z\| < \delta/2, \tag{3.11}$$

where

$$W(t) = \text{diag}(W_0(t), \overbrace{1, 1, \dots, 1}^n),$$

and

$$z = \text{diag}(z_0, \omega, \omega^2, \dots, \omega^n).$$

and where $\omega = e^{2\pi i/n+1}$.

We assert that one can find unitaries V_0, V_1, \dots, V_m such that

$$\|V_i z V_i^* - W(t_i)z\| < \Delta_{z_0}(\varepsilon, 2m+1-2i) \quad \text{for any } 0 \leq i \leq m,$$

and

$$\text{bott}_1(V_{i+1}^* V_i, z) = 0 \quad \text{for any } 0 \leq i \leq m-1.$$

Assume that the unitaries V_0, V_1, \dots, V_i satisfy the condition above. As indicated above, one can find a unitary V_{i+1} such that

$$\|V_{i+1} z V_{i+1}^* - W(t_{i+1})z\| < \min\{\frac{1}{2}\delta_e(\frac{1}{2}\Delta_{z_0}(\varepsilon, 2m-2i)), \frac{1}{2}\Delta_{z_0}(\varepsilon, 2m-2i-1)\},$$

and denote

$$b_i := \text{bott}_1(V_{i+1}^* V_i, z).$$

By Theorem 3.6 of [15], one has that for any $\tau \in T(A)$

$$\begin{aligned} |\tau(b_i)| &= \frac{1}{2\pi} |\tau(\log(z^* V_{i+1}^* V_i z V_i^* V_{i+1}))| \\ &= \frac{1}{2\pi} |\tau(\log(V_{i+1} z^* V_{i+1}^* V_i z V_i^*))|. \end{aligned}$$

Note that

$$\begin{aligned} \left\| V_{i+1} z^* V_{i+1}^* V_i z V_i^* - z^* W^*(t_{i+1}) W(t_i) z \right\| &< \frac{1}{2} \delta_\varepsilon \left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \right) + \Delta_{z_0}(\varepsilon, 2m-2i+1) \\ &< \delta_\varepsilon \left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \right). \end{aligned}$$

Therefore, we have

$$\left\| \log(V_{i+1} z^* V_{i+1}^* V_i z V_i^*) - \log(z^* W^*(t_{i+1}) W(t_i) z) \right\| < \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i),$$

and hence

$$\begin{aligned} |\tau(b_i)| &< \frac{1}{2\pi} |\tau(\log(W^*(t_{i+1}) W(t_i)))| + \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \\ &= \frac{1}{2\pi} (t_{i+1} - t_i) |\tau(p) - \tau(q)| + \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \\ &< \Delta_{z_0}(\varepsilon, 2m-2i) < \delta_{z_0} \left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) \right). \end{aligned}$$

Since A has Property (B1), there is a unitary $v'_{i+1} \in A$ such that

$$\left\| v'_{i+1} z_0 (v'_{i+1})^* - z_0 \right\| < \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) \quad \text{and} \quad \text{bott}_1(v'_{i+1}, z_0) = b_i.$$

Set

$$v_{i+1} = \text{diag}(v'_{i+1}, 1, \dots, 1).$$

We then have

$$\left\| v_{i+1} z v_{i+1}^* - z \right\| = \left\| v'_{i+1} z_0 (v'_{i+1})^* - z_0 \right\| < \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1)$$

and

$$\text{bott}_1(v_{i+1}, z) = \text{bott}_1(v'_{i+1}, z_0) = b_i.$$

Therefore

$$\begin{aligned} \left\| (V_{i+1} v_{i+1}) z (v_{i+1}^* V_{i+1}^*) - W(t_{i+1}) z \right\| &< \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) + \left\| V_{i+1} z V_{i+1}^* - W(t_{i+1}) z \right\| \\ &< \Delta_{z_0}(\varepsilon, 2m-2i-1) \end{aligned}$$

and

$$\text{bott}_1((v_{i+1}^* V_{i+1}^*) V_i, z) = \text{bott}_1(v_{i+1}^*, z) + \text{bott}_1(V_{i+1}^* V_i, z) = 0.$$

By replacing V_{i+1} by $V_{i+1} v_{i+1}$, we proved the assertion.

Then, for each V_i and V_{i+1} , there is a path of unitary $V^{(i)}(t)$ such that

$$\| [V^{(i)}(t), z] \| < \varepsilon, \quad V^{(i)}(0) = 1, \quad V^{(i)}(1) = V_{i+1}^* V_i.$$

By setting $U^{(i)}(t) = V_{i+1}V^{(i)}(1-t)$, we have that

$$U^{(i)}(0) = V_i, \quad U^{(i)}(1) = V_{i+1},$$

and for each t ,

$$\begin{aligned} \left\| U^{(i)}(t)z(U^{(i)}(t))^* - W(t)z \right\| &= \left\| V_{i+1}V^{(i)}(1-t)z(V^{(i)}(1-t))^*V_{i+1}^* - W(t)z \right\| \\ &< \varepsilon + \left\| V_{i+1}zV_{i+1}^* - W(t)z \right\| \\ &\leq 2\varepsilon \leq \frac{1}{4}. \end{aligned}$$

By connecting all $U^{(i)}(t)$, we get a path of unitary $U(t)$ such that for any t ,

$$\|U(t)zU^*(t) - W(t)z\| < \frac{1}{4}.$$

In particular,

$$a = [W(t)]_1 = [\mathbf{p}(U, z)].$$

Moreover,

$$\|[U(1), z]\| = \|[V_m, z]\| < \delta^{\mathbf{p}},$$

as desired. \square

Corollary 3.10. *Let A be a unital separable simple C^* -algebra with $\text{TR}(A) = 0$. Then there exists $\delta > 0$ such that for any element $a \in K_0(A)$, if $|\tau(a)| \leq \delta$ for any $\tau \in \text{T}(A)$, then one has*

$$a = [\mathbf{p}(U, z)] \in K_1(\text{S}(A))$$

for some unitary $z \in \text{M}_k(A)$ and some $U(t) \in \text{C}([0, 1], \text{U}(\text{M}_k(A)))$ (for some $k \geq 1$) with $U(0) = 1$ and $\|[U(1), z]\| < \delta^{\mathbf{p}}$.

Proof. By Lemma 5.2 of [15], A has Property (B1). Then, the statement follows from Lemma 3.9. \square

Corollary 3.11. *Let A be a unital separable simple C^* -algebra with $\text{TR}(A) = 0$. Then $C(A) = K_0(A)$.*

Proof. Denote by δ the constant of Corollary 3.10. For any $a \in K_0(A)$, since $K_0(A)$ is tracially approximately divisible (see Definition 3.15), one has

$$a = a_1 + a_2 + \cdots + a_n$$

with $|\tau(a_i)| < \delta$ for each $1 \leq i \leq n$. Therefore, $a_i \in C(A)$. Since $C(A)$ is a group, one has that $a \in C(A)$, as desired. \square

Definition 3.12. For any unitary u in a C^* -algebra A , denote by $R(u, t)$ the unitary path in $\text{M}_2(A)$ defined by

$$R(u, t) := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi t}{2}) & \sin(\frac{\pi t}{2}) \\ -\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi t}{2}) & -\sin(\frac{\pi t}{2}) \\ \sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix}, \quad t \in [0, 1].$$

Note that

$$R(u, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R(u, 1) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

Lemma 3.13. *Let A be a unital C^* -algebra. Let $B \subseteq A$ be a unital separable C^* -subalgebra. Write*

$$K_0(B)_+ = \{k_1, k_2, \dots, k_n, \dots\}$$

and

$$K_1(B) = \{h_1, h_2, \dots, h_n, \dots\},$$

and denote by $K_n = \langle k_1, \dots, k_n \rangle$ the subgroup generated by $\{k_1, \dots, k_n\}$, and by $H_n = \langle h_1, \dots, h_n \rangle$ the subgroup generated by $\{h_1, \dots, h_n\}$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets whose union is dense in B . Assume that for each i , there exist a projection $p_i \in M_{r_i}(\mathcal{F}_i)$ and a unitary $z_i \in M_{r_i}(\mathcal{F}_i)$ with $[p_i]_0 = k_i$ and $[z_i]_1 = h_i$. Let $\{u_n\}$ be a sequence of unitaries such that

$$\|[u_{n+1}^{(r_n)}, a]\| \leq \frac{\delta^n}{2^{n+1}}$$

for any $a \in M_{r_n}(w_n^* \mathcal{F}_n w_n)$, where $w_n = u_1 \cdots u_n$ and $u_k^{(r_j)} = \text{diag}(\overbrace{u_k, u_k, \dots, u_k}^{r_j})$. Then

$$\alpha = \lim_{n \rightarrow \infty} \text{Ad}(w_n)$$

defines a monomorphism from B to A , and the extensions $\eta_0(M_\alpha)$ and $\eta_1(M_\alpha)$ are determined by the inductive limits

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n+1} & \longrightarrow & K_{n+1} \longrightarrow 0 \\ & & \parallel & & \uparrow \gamma_n^0 & & \uparrow \iota_{n,n+1} \\ 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \longrightarrow & K_n \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} \longrightarrow 0 \\ & & \parallel & & \uparrow \gamma_n^1 & & \uparrow \iota_{n,n+1} \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n \longrightarrow 0 \end{array}$$

respectively, where $\gamma_n^0 : K_n \rightarrow K_1(A)$ is defined by $k_i \mapsto [(w_n^* p_i w_n^*) u_{n+1} (w_n^* p_i w_n) + (1 - w_n^* p_i w_n)]$ and $\gamma_n^1 : H_n \rightarrow K_0(A)$ is defined by $h_i \mapsto [\mathbf{p}(R^*(u_{n+1}, t), w_n^* z_i w_n)]$.

Proof. It is clear that we may assume that $r_n \leq r_{n+1}$, $n = 1, 2, \dots$

Note that for any $a \in \mathcal{F}_n$,

$$\begin{aligned} \|\text{Ad}(w_{n+k})(a) - \text{Ad}(w_n)(a)\| &\leq \sum_{i=1}^k \|\text{Ad}(u_1 u_2 \cdots u_{n+i})(a) - \text{Ad}(u_1 u_2 \cdots u_{n+i-1})(a)\| \\ &\leq \sum_{i=1}^k \frac{1}{2^{n+i}} \leq \frac{1}{2^n}. \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \text{Ad}(w_n)$ exists if $n \rightarrow \infty$. Denote by $\alpha_n = \text{Ad}(w_n)$, and denote its limit by α . Note that α is a monomorphism. Moreover,

$$[\alpha] = [\iota] \text{ in } KL(B, A), \quad (3.12)$$

where $\iota : B \rightarrow A$ is the embedding. It follows that the six-term exact sequence in 2.3 splits.

To simplify notation, without loss of generality, in what follows, we may replace $\alpha \otimes \text{id}_{M_{r_i}}$ by α , $\alpha_n \otimes \text{id}_{M_{r_i}}$ by α_n , $u_n^{(r_n)}$ by u_n , and $w_n^{(r_n)}$ by w_n respectively (and write $p_i \in A$).

Consider $K_0(M_\alpha(A))$ first. Fix n , and note that

$$\|\alpha_n(p_i) - \alpha(p_i)\| \leq \frac{1}{4}, \quad \text{for any } 1 \leq i \leq n.$$

Therefore, for each i , there is a unitary v_i with $\|v_i - 1\| \leq \frac{1}{2}$ such that $\alpha(p_i) = v_i^* \alpha_n(p_i) v_i$. In particular, there is a path $r_i^{(n)}(t)$ of projection with $r_i^{(n)}(0) = \alpha_n(p_i)$, $r_i^{(n)}(1) = \alpha(p_i)$, and $\|r_i^{(n)}(t) - \alpha(p_i)\| \leq \frac{1}{2}$. Then there is a homomorphism $\psi_n^{(0)} : K_1(A) \oplus K_n \rightarrow K_0(M_\alpha)$ defined by

$$\psi_n^{(0)} : ([Q(t)], [p_i]) \mapsto [Q(t)] + [P_i^{(n)}(t)],$$

where $P_i^{(n)}(t)$ is the path

$$P_i^{(n)}(t) = \begin{cases} R^*(w_n, 2t)p_i R(w_n, 2t) & 0 \leq t \leq \frac{1}{2} \\ r_i^{(n)}(2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

and $[Q(t)] \in K_0(S(A))$. Since $[\pi]_0 \circ \psi_n^{(0)} = \text{id}_{K_n}$, the map $\psi_n^{(0)}$ is injective.

We then have

$$\psi_{n+1}^{(0)}([Q(t)], [p_i]) - \psi_n^{(0)}([Q(t)], [p_i]) = [P_i^{(n+1)}(t)] - [P_i^{(n)}(t)] \in K_0(S(A)). \quad (3.13)$$

Moreover, it is easy to see that

$$[P_i^{(n+1)}(t)] - [P_i^{(n)}(t)] = [R^*(u_{n+1}, t)w_n^* p_i w_n R(u_{n+1}, t)] - [w_n^* p_i w_n] \in K_0(S(A)).$$

Define homomorphism $\psi_{n,n+1}^{(0)} : K_1(A) \oplus K_n \rightarrow K_1(A) \oplus K_{n+1}$ by

$$\psi_{n,n+1}^{(0)} : (Q(t), [p_i(t)]) \mapsto ([Q(t)] + ([P_i^{(n)}(t)] - [P_i^{(n+1)}(t)]), [p_i(t)]).$$

By the construction we have that $\psi_{n+1}^{(0)} \circ \psi_{n,n+1}^{(0)} = \psi_n^{(0)}$. Thus, we obtain a homomorphism

$$\psi^{(0)} : \varinjlim (K_0(M_{\alpha_n}), \psi_{n,n+1}) \rightarrow K_0(M_\alpha(A)).$$

Let us show that $\psi^{(0)}$ is surjective. For each projection $p(t) \in M_\alpha$, we can assume that $[p(0)]_0 = k_i = [p_i]_0 \in K_n$ for some $i \leq n$, and if denote by v the partial isometry with $v^* p_i v = p(0)$, then $\|w_n^* v w_n - v\| \leq \frac{1}{2}$. Then there is a path of partial isometries $v(t) \in M_\alpha$ such that $v(0) = v$ and $v(1) = \alpha(v)$. Then

$$h := [p(t)] - [v^*(t)P_i^{(n)}(t)v(t)]$$

is an element in $K_0(S(A))$ and $[p(t)]_0 = \psi^{(0)}(h, k_i)$. Therefore $\psi^{(0)}$ is surjective. The injectivity of $\psi^{(0)}$ follows from the injectivity of each $\psi_n^{(0)}$. Thus, $\psi^{(0)}$ is an isomorphism.

Let us show that $\psi_{n,n+1}^{(0)}$ has the desired form.

Consider the invertible element $c = (w_n^* p_i w_n)u_{n+1}(w_n^* p_i w_n) + (1 - (w_n^* p_i w_n)) \in A$ (which is close to a unitary).

Let us calculate the corresponding element in $S(A)$. Consider the path

$$Z(t) = \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} R(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 1 - w_n^* p_i w_n \end{pmatrix}.$$

Note that $Z(0) = \text{diag}(1, 1)$ and $Z(1) = \text{diag}(c, c^*)$ and $Z(t)$ is invertible for any t with $\|Z^*(t)Z(t) - 1\| \leq \frac{1}{2^n}$.

Let

$$e(t) = (Z^*(t)Z(t))^{-\frac{1}{2}} Z^*(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z(t) (Z^*(t)Z(t))^{-\frac{1}{2}}.$$

Then the element in $K_0(S(A))$ which corresponds to c is $[e] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$. However, since

$$\left\| \left[\begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix}, R(u_{n+1}, t) \right] \right\| \leq 2 \| [w_n^* p_i w_n, u_{n+1}] \| \leq \frac{1}{2^n},$$

a direct calculation shows

$$\begin{aligned} e(t) &\approx_{2/(2^n-1)} \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \\ &\approx_{2/2^n} R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} R(u_{n+1}, t) \\ &\quad + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \\ &= R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t) + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and therefore, for $n > 2$,

$$\begin{aligned} [e] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] &= [R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t)] - [\begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix}] \\ &= [R^*(u_{n+1}, t) w_n^* p_i w_n R(u_{n+1}, t)] - [w_n^* p_i w_n] \\ &= [P_i^{(n+1)}(t)] - [P_i^{(n)}(t)]. \end{aligned}$$

Therefore, the corresponding element of $[P_i^{(n+1)}(t)] - [P_i^{(n)}(t)]$ in $K_1(A)$ is

$$(w_n^* p_i w_n) u_{n+1} (w_n^* p_i w_n) + (1 - (w_n^* p_i w_n)),$$

and hence the map $\psi_{n,n+1}^{(0)}$ has the desired form.

Now, let us consider $K_1(M_\alpha)$. For each n , consider the unitaries $\{z_i; 1 \leq i \leq n\}$. Note that

$$\|\alpha(z_i) - \alpha_n(z_i)\| \leq \frac{1}{2^n}$$

for each i . Then, there are paths of unitaries $s_i^{(n)}$ such that $s_i^{(n)}(0) = \alpha_n(z_i)$, $s_i^{(n)}(1) = \alpha(z_i)$, and

$$\|s_i^{(n)}(t) - \alpha(z_i)\| \leq \frac{1}{2^{n-1}}$$

for each $t \in [0, 1]$. Define a homomorphism $\psi_n^{(1)} : K_0(A) \oplus H_n \rightarrow K_1(M_\alpha(A))$ by

$$\psi_n^{(1)} : ([S(t)], z_i) \mapsto [S(t)] + [V_i^{(n)}(t)]$$

where $V_i^{(n)}$ is the path

$$V_i^{(n)}(t) = \begin{cases} R^*(w_n, 2t)z_i R(w_n, 2t) & 0 \leq t \leq \frac{1}{2} \\ s_i^{(n)}(2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

and $S(t)$ is a unitary in $U_\infty(\widetilde{S(A)})$.

We then have

$$\begin{aligned} \psi_{n+1}([S(t)], [v_i]) - \psi_n([S(t)], [v_i]) &= [V_i^{(n+1)}(t)] - [V_i^{(n)}(t)] \\ &= [\mathbf{p}(R^*(u_{n+1}, t), w_n^* v_i w_n)] \in K_1(S(A)). \end{aligned}$$

Define homomorphism $\psi_{n,n+1}^{(1)} : K_0(A) \oplus H_n \rightarrow K_0(A) \oplus H_{n+1}$ by

$$\psi_{n,n+1}^{(0)} : (S(t), [v_i]) \mapsto ([S(t)] + ([V_i^{(n)}(t)] - [V_i^{(n+1)}(t)]), [v_i]).$$

By the construction we have that $\psi_{n+1}^{(1)} \circ \psi_{n,n+1}^{(1)} = \psi_n^{(1)}$. Thus, there is a homomorphism

$$\psi^{(1)} : \varinjlim (K_0(M_{\alpha_n}), \psi_{n,n+1}) \rightarrow K_0(M_\alpha).$$

By the same argument as that for $K_0(M)_\alpha$, the map $\psi^{(1)}$ is an isomorphism, and moreover, it has the desired form. \square

3.14. Let

$$0 \longrightarrow G \longrightarrow E \xrightarrow{\pi} H \longrightarrow 0$$

be a short exact sequence of abelian groups with H countable. We assume that the extension is pure, i.e., for any finitely generated subgroup $H' \subseteq H$, there is a homomorphism $\theta' : H' \rightarrow E$ such that $\pi \circ \theta' = \text{id}_{H'}$.

Write $H = \{h_1, h_2, \dots\}$. Consider $H_n = \langle h_1, \dots, h_n \rangle$, the subgroup generated by $\{h_1, \dots, h_n\}$. Since E is a pure extension, there is a map $\theta_n : H_n \rightarrow E$ such that $\pi \circ \theta_n = \text{id}_{H_n}$. Let us call this map θ_n a partial splitting map. Denote by ι_n the inclusion map from H_n to H_{n+1} , and set the map $\gamma_n : H_n \rightarrow E$ by

$$\gamma_n = \theta_{n+1} \circ \iota_n - \theta_n.$$

It is clear that $\gamma_n(H_n) \subseteq G \subseteq E$. Therefore, let us regard γ_n as a map from H_n to G .

Definition 3.15. A partially ordered group G with an order unit is tracially approximately divisible if for any $a \in G$, any $\varepsilon > 0$, and any natural number n , there exist $b \in G$ such that $|\tau(a - nb)| \leq \varepsilon$ for any state τ of G .

Lemma 3.16. *With the setting as 3.14, if, in addition, G is a partially ordered group with an order unit which is tracially approximately divisible, then, for any $\varepsilon > 0$ and any given partial splitting map θ_n with $\pi \circ \theta_n = \text{id}_{H_n}$, one can choose a partial splitting map θ_{n+1} such that $|\tau(\gamma_n(h_i))| < \varepsilon$ for any $1 \leq i \leq n$ and any state τ of G .*

Proof. Pick any partial splitting map $\theta'_{n+1} : H_{n+1} \rightarrow E_1$, and consider the map $\gamma'_n : H_n \rightarrow G$ defined by

$$\gamma'_n = \theta'_{n+1} \circ \iota_n - \theta_n.$$

Consider the \mathbb{Q} -linear map

$$\gamma'_n \otimes \text{id} : H_n \otimes \mathbb{Q} \rightarrow G \otimes \mathbb{Q}.$$

Then it has an extension to $H_{n+1} \otimes \mathbb{Q}$. That is, there is a linear map $\psi : H_{n+1} \otimes \mathbb{Q} \rightarrow G \otimes \mathbb{Q}$ such that

$$\begin{array}{ccc} G \otimes \mathbb{Q} & \xleftarrow{\psi} & H_{n+1} \otimes \mathbb{Q} \\ & \nwarrow \gamma'_n \otimes \text{id} & \uparrow \iota_n \otimes \text{id} \\ & & H_n \otimes \mathbb{Q} \end{array}$$

commutes.

Since H_{n+1} is finitely generated, $H_{n+1} \cong \mathbb{Z}^{k_{n+1}} \oplus T_{n+1}$ for some finite abelian group T_{n+1} . Denote by $\{e_1, e_2, \dots, e_{k_{n+1}}\}$ the standard generators for the torsion free part of H_{n+1} . Then, for each $1 \leq i \leq k_{n+1}$, we have

$$\psi(e_i) = \sum_{j=1}^{l_i} r_j^{(i)} g_j^{(i)}$$

for some $r_j^{(i)} \in \mathbb{Q}$ and $g_j^{(i)} \in G$.

Write $(\iota_n(h_s))_{\text{free}} = \sum_i m_i^{(s)} e_i$, and denote by $m = \max\{m_i^{(s)}; 1 \leq s \leq n, 1 \leq i \leq k_{n+1}\}$. Since G is tracially approximately divisible, for each e_i , one can find $p_i \in G$ such that

$$\tau(\psi(e_i) - p_i) < \frac{\varepsilon}{k_{n+1}m}$$

for any state τ of G .

Define the map $\phi : H_{n+1} \rightarrow K_0(A) \subset E_1$ by sending e_i to p_i and T_{n+1} to $\{0\}$, and let us consider the map

$$\theta_{n+1} := \theta'_{n+1} - \phi.$$

Then, the map θ_{n+1} satisfies the lemma. Indeed, it is clear that $\pi \circ \theta_{n+1} = \text{id}_{H_{n+1}}$. Moreover, for any h_s and any τ , we have

$$\begin{aligned} |\tau(\theta_{n+1} \circ \iota_n(h_s) - \theta_n(h_s))| &= |\tau(\theta'_{n+1} \circ \iota_n(h_s) - \phi \circ \iota_n(h_s) - \theta_n(h_s))| \\ &= |\tau(\gamma'_n(h_s) - \phi \circ \iota_n(h_s))| \\ &= |\tau(\psi \circ \iota_n(h_s) - \phi \circ \iota_n(h_s))| \\ &= \left| \sum_i m_i^{(s)} \tau(\psi(e_i) - p_i) \right| \\ &< \varepsilon, \end{aligned}$$

as desired. □

Theorem 3.17. *Let A be a unital simple C^* -algebra and let $B \subseteq A$ be a unital separable C^* -subalgebra. Suppose that A contains a positive element b with $\text{sp}(b) = [0, 1]$, and A has Property (B1) and Property (B2) associated with B and Δ_B and $K_0(A)$ is tracially approximately divisible. For any $E_0 \in \text{Pext}(K_1(A), K_0(B))$ and $E_1 \in \text{Pext}(K_0(A), K_1(B))$, there exists $\alpha \in \overline{\text{Im}}(B, A)$ such that $\eta_0(M_\alpha) = E_0$ and $\eta_1(M_\alpha) = E_1$.*

Proof. Write

$$K_0(B)_+ = \{k_1, k_2, \dots, k_n, \dots\}$$

and

$$K_1(B) = \{h_1, h_2, \dots, h_n, \dots\},$$

and consider the subgroups $K_n := \langle k_1, \dots, k_n \rangle$ and $H_n := \langle h_1, \dots, h_n \rangle$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets in the unit ball of B with the union dense in the unit ball of B . Assume that for each i , there is a projection $p_i \in M_{r_i}(\mathcal{F}_i)$ and unitary $z_i \in M_{r_i}(\mathcal{F}_i)$ with $[p_i]_0 = k_i$ and $[z_i]_1 = h_i$. We may assume that $r_i \leq r_{i+1}$, $i \in \mathbb{N}$.

We assert that there are unitaries $\{u_n\}$ and diagrams

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1(A) & \longrightarrow & E_0 & \xrightarrow{\pi} & K_0(B) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
& & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n+1} & \xrightarrow[\theta_{n+1}^0]{\pi_n} & K_{n+1} \longrightarrow 0 \\
& & \parallel & & \uparrow \swarrow \gamma_n^0 & & \uparrow \iota_{n,n+1} \\
& & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \xrightarrow[\theta_n^0]{\pi} & K_n \longrightarrow 0 \\
& & \parallel & & \uparrow \swarrow \gamma_n^0 & & \uparrow \iota_{n,n+1} \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_0(A) & \longrightarrow & E_1 & \xrightarrow{\pi} & K_0(B) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
& & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \xrightarrow[\theta_{n+1}^1]{\pi_n} & H_{n+1} \longrightarrow 0 \\
& & \parallel & & \uparrow \swarrow \gamma_n^1 & & \uparrow \iota_{n,n+1} \\
& & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \xrightarrow[\theta_n^1]{\pi} & H_n \longrightarrow 0 \\
& & \parallel & & \uparrow \swarrow \gamma_n^1 & & \uparrow \iota_{n,n+1} \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

such that

$$\|[u_{n+1}, a]\| \leq \frac{\delta^{\mathfrak{p}}}{r_n^2 \cdot 2^{n+1}}$$

for any $a \in M_{r_n}(w_n^* \mathcal{F}_n w_n)$, where $w_n = u_1 \cdots u_n$ and $u_1 = 1$. The image of each γ_n^1 lies inside $C(A)$ so that $\Lambda \circ \gamma_n^1$ is well-defined, and

$$\text{bott}_1(w_n^* z_i w_n, u_{n+1}) = \Lambda \circ \gamma_n^1(h_i) \quad \text{and} \quad \text{bott}_0(w_n^* p_i w_n, u_{n+1}) = \gamma_n^0(k_i).$$

Moreover, each partial splitting map θ_n^i ($i = 0, 1$) can be extended to a partial splitting map $\tilde{\theta}_n^i$ ($i = 0, 1$) defined on the subgroup generated by $K_n \cup \mathcal{G}_0^n$ or $H_n \cup \mathcal{G}_1^n$, where \mathcal{G}_i ($i = 0, 1$) is the set of generators of G_i

($i = 0, 1$) of Definition 3.6 with respect to $\frac{\delta^p}{r_{n+1}^2 \cdot 2^{n+1}}$, \mathcal{F}_n , $\mathcal{P}_0^{(n)}$, $\mathcal{P}_1^{(n)}$, and ι , where $\mathcal{P}_0 = \{[p_1], [p_2], \dots, [p_n]\}$ and $\mathcal{P}_1 = \{[z_1], [z_2], \dots, [z_n]\}$. Denote by the subgroups generated by $K_n \cup \mathcal{G}_0$ and $H_n \cup \mathcal{G}_1$ by \tilde{K}_n and \tilde{H}_n respectively.

Assume that we have constructed the unitaries $\{u_1 = 1, u_2, \dots, u_n\}$ and the diagrams

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1(A) & \longrightarrow & E_0 & \xrightarrow{\pi} & K_1(B) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \xleftarrow[\theta_n^0]{\pi_n} & K_n \longrightarrow 0 \\
& & \parallel & & \nwarrow \gamma_{n-1}^0 & \nearrow \iota_{n-1,n} & \uparrow \iota_{n-1,n} \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n-1} & \xleftarrow[\theta_{n-1}^0]{\pi} & K_{n-1} \longrightarrow 0 \\
& & \parallel & & \nwarrow & \nearrow & \uparrow \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_0(A) & \longrightarrow & E_1 & \xrightarrow{\pi} & K_0(B) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \xleftarrow[\theta_n^1]{\pi_{n-1}} & H_n \longrightarrow 0 \\
& & \parallel & & \nwarrow \gamma_{n-1}^1 & \nearrow \iota_{n-1,n} & \uparrow \iota_{n-1,n} \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n-1} & \xleftarrow[\theta_{n-1}^1]{\pi} & H_{n-1} \longrightarrow 0 \\
& & \parallel & & \nwarrow & \nearrow & \uparrow \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

satisfying the above assertion.

Denote by

$$\delta_n = \Delta_B\left(\frac{\delta^p}{r_{n+1}^2 \cdot 2^{n+1}}, \mathcal{F}_n, \mathcal{P}_0^{(n)}, \mathcal{P}_1^{(n)}, \text{ad}(w_n) \circ \iota\right),$$

where $\mathcal{P}_0 = \{[p_1], [p_2], \dots, [p_n]\}$ and $\mathcal{P}_1 = \{[z_1], [z_2], \dots, [z_n]\}$. Set

$$w_n^{(r_n)} = \text{diag}(\overbrace{w_n, w_n, \dots, w_n}^{r_n}), \quad n = 1, 2, \dots$$

We note that $[p_i] = [(w_n^{(r_n)})^* p_i w_n^{(r_n)}]$ and $[z_i] = [(w_n^{(r_n)})^* z_i w_n^{(r_n)}]$, $i = 1, 2, \dots$

Since E_0 and E_1 are pure extensions, there are partial splitting maps $\tilde{\theta}_{n+1}^0 : \tilde{K}_{n+1} \rightarrow E_0$ and $\tilde{\theta}_{n+1}^1 : \tilde{H}_{n+1} \rightarrow E_1$. Since $K_0(A)$ is tracially approximately divisible, by Lemma 3.16, the partial splitting map $\tilde{\theta}_{n+1}^1$ can be chosen so that for any $g \in \mathcal{G}_i^n \cup \{g_1, \dots, g_n\}$,

$$|\tau(\gamma_n^1(g))| \leq \min\{\delta, \delta_n\}, \quad \text{for all } \tau \in T(A),$$

where δ is the constant of Lemma 3.9 (since A has Property (B1)). Note that the maps γ_n^0 and γ_n^1 are defined on \tilde{K}_n and \tilde{H}_n respectively, in particular, on G_0 and G_1 respectively.

By Lemma 3.9, one has

$$\gamma_n^1(h_i) = [U^*(t)zU(t)z^*]_1 \in K_1(S(A))$$

for a unitary $z \in A$ and a path $U(t) \in C([0, 1], U_\infty(A))$ with $U_0 = 1$ and $\|[U(1), z]\| \leq \delta^p$. Therefore, the map

$$\Lambda \circ \gamma_n^1|_{H_n} : H_n \rightarrow K_0(A)$$

is well-defined, and by Lemma 3.5,

$$|\tau(\Lambda \circ \gamma_n^1(h_i))| = |\tau(\gamma_n^1(h_i))| \leq \delta_n$$

for any $\tau \in T(A)$ and $1 \leq i \leq n$.

Put $b_0 = \gamma_n^0$ and $b_1 = \Lambda \circ \gamma_n^1$. By the assumption that A has Property (B2) associated with B and Δ_B , there is a unitary $u_{n+1} \in A$ such that

$$\|[u_{n+1}, a]\| \leq \frac{1}{r_n^2 \cdot 2^{n+1}}$$

for any $a \in M_{r_n}(w_n^* \mathcal{F}_n w_n)$ and

$$\text{bott}_1(w_n^* z_i w_n, u_{n+1}) = \Lambda \circ \gamma_n^1(h_i) \quad \text{and} \quad \text{bott}_0(w_n^* p_i w_n, u_{n+1}) = \gamma_n^0(k_i).$$

Denote by θ_{n+1}^i ($i = 0, 1$) the restriction of $\tilde{\theta}_{n+1}^i$ ($i = 0, 1$) to K_{n+1} and H_{n+1} respectively. Repeating this procedure, we get a sequence of unitaries $\{u_n\}$ and diagrams satisfying the assertion.

By Lemma 3.13, the inner automorphisms $\{\text{Ad}(u_1 \cdots u_n)\}$ converge to a monomorphism α , and the extension $\eta_0(M_\alpha)$ and $\eta_1(M_\alpha)$ are determined by the inductive limits of

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n+1} & \longrightarrow & K_{n+1} \longrightarrow 0 \\ & & \parallel & & \nwarrow \tilde{\gamma}_n^0 & \nearrow \iota_{n,n+1} & \nearrow \iota_{n,n+1} \\ 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \longrightarrow & K_n \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} \longrightarrow 0 \\ & & \parallel & & \nwarrow \tilde{\gamma}_n^1 & \nearrow \iota_{n,n+1} & \nearrow \iota_{n,n+1} \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n \longrightarrow 0 \end{array}$$

respectively, where

$$\tilde{\gamma}_n^0(k_i) = \tilde{\gamma}_n^0([p_i]) = [(w_n^* p_i w_n) u_{n+1} (w_n^* p_i w_n) + (1 - w_n^* p_i w_n)]$$

and

$$\tilde{\gamma}_n^1(h_i) = \tilde{\gamma}_n^1([z_i]) = [\mathbf{p}(R^*(u_{n+1}, t), w_n^* z_i w_n)],$$

and therefore,

$$\tilde{\gamma}_n^0(k_i) = \text{bott}_0(w_n^* p_i w_n, u_{n+1}) = \gamma_n^0(k_i), \quad 1 \leq i \leq n,$$

that is, $\tilde{\gamma}_n^0 = \gamma_n^0$. For each cross map $\tilde{\gamma}_n^1$, one has

$$\Lambda \circ \tilde{\gamma}_n^1(h_i) = \text{bott}(w_n^* z_i w_n, R(u_{n+1}, 1)) = \text{bott}(w_n^* z_i w_n, u_{n+1}) = \Lambda \circ \gamma_n^1(h_i), \quad 1 \leq i \leq n.$$

Since Λ is injective, we have that $\tilde{\gamma}_n^1 = \gamma_n^1$. Hence, one has that $\eta_0(M_\alpha) = E_0$ and $\eta_1(M_\alpha) = E_1$, as desired. \square

Proposition 3.18. *Let C be a unital AH-algebra and let A be a unital separable simple C^* -algebra with $\text{TR}(A) = 0$. Suppose that there is a unital monomorphism $h : C \rightarrow A$. Then A has Property (B2) associated with C for some Δ_C as described in 3.6.*

Proof. Fix $\varepsilon > 0$, finite subset $\mathcal{F} \subset C$, $\mathcal{P}_0 \subset K_0(C)$ and $\mathcal{P}_1 \subset K_1(C)$. Write $C = \varinjlim (C_n, \psi_n)$ so that C_n and ψ_n satisfy the conditions in 7.2 of [15]. Let $\Delta_B(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h)$ be δ as required by Lemma 7.5 of [15] for the above $\varepsilon, \mathcal{F}, \mathcal{P}$. Let $n \geq 1$ be an integer in 7.5 of [15] so we may assume that

$$\mathcal{P} \subset \cup_{i=0,1} (\psi_{n,\infty})_{*i}(K_i(C_n))$$

and let $k(n) \geq n$ be as in 7.5 of [15].

Put $G_i = (\psi_{k(n),\infty})_{*i}(K_i(C_{k(n)}))$, $i = 0, 1$. In particular, G_i is finitely generated and $\mathcal{P}_i \subset G_i$, $i = 0, 1$.

Let $b_i : G_i \rightarrow K_{i-1}(A)$ be given, $i = 0, 1$. Write

$$K_i(C_{k(n)} \otimes C(\mathbb{T})) = K_i(C_{k(n)}) \oplus \beta(K_{i-1}(C_{k(n)})), \quad i = 0, 1 \quad \text{and} \quad (3.14)$$

$$\underline{K}(C_{k(n)} \otimes C(\mathbb{T})) = \underline{K}(C_{k(n)}) \oplus \beta(\underline{K}(C_{k(n)})) \quad (3.15)$$

(see 2.10 of [14]).

Define $\kappa^{(0)} : \underline{K}(C_{k(n)}) \rightarrow \underline{K}(A)$ by $\kappa^{(0)} = [h \circ \psi_{k(n),\infty}]$. Define $\kappa_i^{(1)} : \beta(K_i(C_{k(n)})) \rightarrow K_{i-1}(A)$ by

$$\kappa_i^{(1)} \circ \beta(x) = b_{i-1}(x)$$

for $x \in K_{i-1}(C_{k(n)})$, $i = 0, 1$. Since C satisfies the UCT, there is $\kappa^{(1)} \in KK(S(C_{k(n)}), A)$ such that $\Gamma(\kappa^{(1)}) = \kappa_i^{(1)}$. Define $\kappa : \underline{K}(C_{k(n)} \otimes C(\mathbb{T})) \rightarrow \underline{K}(A)$ by $\kappa|_{\underline{K}(C_{k(n)})} = \kappa^{(0)}$ and $\kappa|_{\beta(\underline{K}(C_{k(n)}))} = \kappa^{(1)}$.

The lemma then follows from Lemma 7.5 of [15]. \square

Lemma 3.19. *Let B be a unital separable simple amenable C^* -algebra with $\text{TR}(B) = 0$ which satisfies the UCT, and let A be a unital simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Suppose that $\bar{\kappa} \in KL(B, A)^{++}$ with $\bar{\kappa}([1_B]) \leq [1_A]$ in $K_0(A)$. Then there is a monomorphism $\alpha : B \rightarrow A$ such that*

$$[\alpha] = \bar{\kappa} \text{ in } KL(B, A). \quad (3.16)$$

Proof. It follows from the classification theorem ([12]) that B is a unital simple AH-algebra with slow dimension growth and with real rank zero. Then the lemma follows from Theorem 4.6 of [10] immediately. \square

Theorem 3.20. *Let A and B be unital simple C^* -algebras with $\text{TR}(A) = 0$ and $\text{TR}(B) = 0$. Assume that B is separable amenable and satisfies the UCT. Then, for any $\kappa \in KK(B, A)^{++}$ with $\kappa([1_B]) \leq [1_A]$ in $K_0(A)$, there is a monomorphism $\alpha : B \rightarrow A$ such that $[\alpha] = \kappa$ in $KK(B, A)$.*

Proof. Denote by $\bar{\kappa}$ the image of κ in $KL(B, A)^+$. It follows from Lemma 3.19 that there exists $\alpha_1 \in \text{Hom}(B, A)$ such that $[\alpha_1]_{KL} = \bar{\kappa}$. By considering the cut-down of A by $\alpha_1(1_B)$, we can regard B as a unital C^* -subalgebra of A with embedding α_1 . Since $\kappa - [\alpha_1]_{KK} \in \text{Pext}(K_*(B), K_{*+1}(A))$, by Theorem 3.17, there is an approximately inner monomorphism α_2 of from B to A such that $[\alpha_2 \circ \alpha_1]_{KK} - [\alpha_1]_{KK} = \kappa - [\alpha_1]_{KK}$. Then, $\alpha := \alpha_2 \circ \alpha_1$ is the desired homomorphism. \square

Let us recall the following theorem from [18]:

Theorem 3.21 (Theorem 5.2 of [18]). *Let C be a unital AH-algebra and let A be a unital simple C^* -algebra with $\text{TR}(A) = 0$. Suppose that there is a pair $\kappa \in KL_e(C, A)^{++}$ and a continuous affine map $\lambda : T(A) \rightarrow T_{\mathbf{f}}(C)$ which is compatible with κ . Then there exists a unital monomorphism $h : C \rightarrow A$ such that*

$$[h] = \kappa \text{ and } h_T = \lambda.$$

Theorem 3.22. *Let C be a unital AH-algebra and let A be a unital simple C^* -algebra with $\text{TR}(A) = 0$. Suppose that $(\kappa, \lambda) \in KKT(C, A)^{++}$. Then there is a monomorphism $\phi : C \rightarrow A$ such that $[\phi] = \kappa$ and $\phi_T = \lambda$.*

Proof. It follows from Theorem 3.21 that there exists a unital monomorphism $\psi : C \rightarrow A$ such that

$$[\psi]_{KL} = \bar{\kappa} \text{ in } KL(C, A) \text{ and } \psi_T = \lambda.$$

Then $\kappa - [\psi] \in \text{Pext}(K_*(C), K_*(A))$. As in the proof of 3.20, we obtain a unital monomorphism $\alpha : \psi(C) \rightarrow A$ for which

$$[\alpha \circ \psi] = [\kappa] \text{ in } KK(C, A)$$

and there exists a sequence of unitaries $\{u_n\} \subset U(A)$ such that

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ \phi(c) = \alpha \circ \phi(c) \text{ for all } c \in C.$$

Put $\phi = \alpha \circ \psi$. Then

$$\lambda(\tau)(c) = \tau \circ \psi(c) = \lim_{n \rightarrow \infty} \tau(\text{ad } u_n \circ \psi(c)) \quad (3.17)$$

$$= \tau(\alpha \circ \psi(c)) = \tau \circ \phi(c) \text{ for all } c \in C. \quad (3.18)$$

It follows that

$$\phi_T = \lambda,$$

as desired. \square

Remark 3.23. It was shown in [18] that there are compact metric space X , unital simple AF-algebras A and $\bar{\kappa} \in KL_e(C(X), A)^{++}$ for which there is no unital homomorphism $h : C(X) \rightarrow A$ so that $[h] = \bar{\kappa}$. Thus the information on λ is essential in general. In the case that C is also simple and has real rank zero, then the map λ is completely determined by $\bar{\kappa}$ since $\rho_C(K_0(C))$ is dense in $\text{Aff}(T(C))$ and $T(C) = T_{\mathbf{f}}(C)$. If the C^* -algebra C is real rank zero and exact, without assuming the simplicity, then $T(C) = S(K_0(C))$, and hence one can define the map $r : T(A) \rightarrow T(C)$ by factoring through $S(K_0(A))$. It is obviously that r is compatible with $\bar{\kappa}$. Moreover, $r(\tau)$ is faithful on C for any $\tau \in T(A)$. Indeed, if $r(\tau)(c) = 0$ for some nonzero positive element $c \in C$, then $r(\tau)(\overline{cC}c) = \{0\}$. In particular, $r(\tau)(p) = 0$ for some nonzero projection $p \in \overline{cC}c$, which contradicts the strict positivity of $\bar{\kappa}$.

Lemma 3.24. *Let X be a Banach space, and let $\{\alpha_n\}$ be a sequence of isometries. If each α_n is invertible, and $\lim_{n \rightarrow \infty} \alpha_n(x)$ and $\lim_{n \rightarrow \infty} \alpha_n^{-1}(x)$ exist for any $x \in X$, then $\alpha := \lim_{n \rightarrow \infty} \alpha_n$ is invertible.*

Proof. Denote by $\beta = \lim_{n \rightarrow \infty} \alpha_n^{-1}$. It is clear that α and β are isometries. Fix an element $x \in X$. For any $\varepsilon > 0$, there exists N such that

$$\|\beta(x) - \alpha_n^{-1}(x)\| \leq \varepsilon \quad \text{and} \quad \|\alpha \circ \beta(x) - \alpha_n \circ \beta(x)\| \leq \varepsilon$$

for any $n \geq N$. Then

$$\begin{aligned} \|\alpha \circ \beta(x) - x\| &\leq \|\alpha \circ \alpha_n^{-1}(x) - x\| + \|\alpha \circ \beta(x) - \alpha \circ \alpha_n^{-1}(x)\| \\ &\leq \|\alpha \circ \alpha_n^{-1}(x) - \alpha_n \circ \alpha_n^{-1}(x)\| + \varepsilon \\ &\leq \|\alpha \circ \alpha_n^{-1}(x) - \alpha \circ \beta(x)\| + \|\alpha \circ \beta(x) - \alpha_n \circ \beta(x)\| + \|\alpha_n \circ \beta(x) - \alpha_n \circ \alpha_n^{-1}(x)\| + \varepsilon \\ &\leq 4\varepsilon. \end{aligned}$$

Since ε is arbitrary, one has that $\alpha \circ \beta(x) = x$, and hence $\alpha \circ \beta = \text{id}$. The argument same as above shows that $\beta \circ \alpha = \text{id}$. Therefore, α is invertible, as desired. \square

Definition 3.25. Let A be a unital C^* -algebra. Denote by $KK_e^{-1}(A, A)^{++}$ the set of those elements $\kappa \in KK_e(A, A)^{++}$ such that κ induces an ordered isomorphism between $K_0(A)$ and isomorphisms between $\underline{K}(A)$.

Corollary 3.26. Let A be a unital separable amenable simple C^* -algebra with $\text{TR}(A) = 0$ and satisfies the UCT. Then, for any $\kappa \in KK_e^{-1}(A, A)^{++}$, there exists an automorphism $\alpha \in \text{Aut}(A)$ such that

$$[\alpha] = \kappa.$$

Proof. By Theorem 3.20 and its remark, there is a monomorphism $\alpha = \alpha_2 \circ \alpha_1$ such that $[\alpha] = \kappa$ in $KK(A, A)$. Let us show that α can be chosen to be an automorphism. Let us first show that α_1 can be chosen to be an automorphism. We first choose a unital monomorphism $\alpha'_1 : A \rightarrow A$ so that $[\alpha'_1] = \kappa$ by 3.20 and its remark.

By the UCT, there is $\kappa_1 \in KK_e^{-1}(A, A)^{++}$ such that $\kappa_1 \times \kappa = \kappa \times \kappa_1 = [\text{id}_A]$. So there is a unital monomorphism $\beta : A \rightarrow A$ such that $[\beta] = \kappa_1$. Then $[\beta \circ \alpha'_1] = [\alpha'_1 \circ \beta] = [\text{id}_A]$. By the uniqueness theorem (2.3 of [11]), both $\beta \circ \alpha'_1$ and $\alpha'_1 \circ \beta$ are approximately unitarily equivalent to the identity. Then, by a standard intertwining argument of Elliott, one obtains an isomorphism $\alpha_1 : A \rightarrow A$ which is approximately unitarily equivalent to α'_1 (see for example Theorem 3.6 of [11]). So in particular, $[\overline{\alpha_1}] = \overline{\kappa}$ in $KL(A, A)$. (This also follows from the proof of Theorem 3.7 of [11] that there is an isomorphism $\alpha_1 : A \rightarrow A$ such that $[\alpha_1] = \overline{\kappa}$ in $KL(A, A)$, using that fact that A is pre-classifiable in the sense of [11], see Theorem 4.2 of [12]).

Consider the map α_2 . Note that if $A = B$, then, in the proof of Theorem 3.17, the union of finite subsets \mathcal{F}_n is dense in A , and the inner automorphisms $\{\text{Ad}(u_1 \cdots u_n)\}$ satisfy Lemma 3.24. Therefore, by Lemma 3.24, the monomorphism $\alpha_2 = \lim_{n \rightarrow \infty} \text{Ad}(u_1 \cdots u_n)$ is an automorphism of A .

Therefore, the map $\alpha = \alpha_2 \circ \alpha_1$ is an automorphism of A . \square

4 Rotation maps

Lemma 4.1. Let H be a finitely generated abelian group, and let A be a C^* -algebra with $\rho_A(K_0(A))$ dense in $\text{Aff}(T(A))$. Let $\psi \in \text{Hom}(H, \text{Aff}(T(A)))$. Fix $\{g_1, \dots, g_n\} \subseteq H$. Then, for any $\varepsilon > 0$, there is a homomorphism $h : H \rightarrow K_0(A)$ such that

$$|\psi(g_i) - \rho_A(h(g_i))| < \varepsilon$$

for any $1 \leq i \leq n$.

Proof. Since the map ψ factors through H/H_{Tor} , we may assume that $H = \bigoplus^k \mathbb{Z}$ for some k and prove the lemma for an $\{g_i\}$ replaced by the standard base $\{e_i; i = 1, \dots, k\}$. Since $\rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$, there exist $a_1, \dots, a_n \in K_0(A)$ such that

$$|\psi(e_i) - \rho_A(a_i)| < \varepsilon, \quad i = 1, \dots, k.$$

The map $h : H \rightarrow K_0(A)$ defined by

$$e_i \mapsto a_i$$

satisfies the lemma. \square

Theorem 4.2. *Let A be a unital simple C^* -algebra with $\rho_A(K_0(A))$ dense in $\text{Aff}(T(A))$. Assume that there is a positive element $b \in A$ with $\text{sp}(b) = [0, 1]$. Let $B \subseteq A$ be a unital C^* -subalgebra and denote by ι the inclusion map. Suppose that A has Property (B1) and Property (B2) associated with B and Δ_B . For any $\psi \in \text{Hom}(K_1(B), \text{Aff}(T(A)))$, there exists $\alpha \in \overline{\text{Inn}}(B, A)$ such that there are maps $\theta_i : K_i(B) \rightarrow K_i(M_\alpha)$ with $\pi_0 \circ \theta_i = \text{id}_{K_i(B)}$, $i = 0, 1$, and the rotation map $R_{\iota, \alpha} : K_1(M_\alpha) \rightarrow \text{Aff}(T(A))$ is given by*

$$R_{\iota, \alpha}(c) = \rho_A(c - \theta_1([\pi_0]_1(c))) + \psi([\pi_0]_1(c)), \quad \forall c \in K_1(M_\alpha(A)).$$

In other words,

$$[\alpha] = [\iota]$$

in $KK(B, A)$, and the rotation map $R_{\iota, \alpha} : K_1(M_\alpha) \rightarrow \text{Aff}(T(A))$ is given by

$$R_{\iota, \alpha}(a, b) = \rho_A(a) + \psi(b)$$

for some identification of $K_1(M_\alpha)$ with $K_0(A) \oplus K_1(B)$.

Proof. Write

$$K_1(B) = \{g_1, g_2, \dots, g_n, \dots\},$$

$$K_0(B)_+ = \{k_1, k_2, \dots, k_n, \dots\},$$

and denote by $H_n = \langle g_1, \dots, g_n \rangle$, the subgroup generated by $\{g_1, \dots, g_n\}$, and $K_n = \langle k_1, \dots, k_n \rangle$, the subgroup generated by $\{k_1, \dots, k_n\}$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets with the union dense in B . Assume that for each i , there is a unitary z_i and a projection p_i in $M_{r_i}(\mathcal{F}_i)$ for some natural number r_i such that $[z_i]_1 = g_i$ and $[p_i] = k_i$. Without loss of generality, we may assume that $r_i \leq r_{i+1}$, $i \in \mathbb{N}$. In what follows, if $v \in A$, by $v^{(m)}$, we mean $v^{(m)} = \text{diag}(\overbrace{v, v, \dots, v}^m)$.

We assert that there are maps $\{h_i : H_i \rightarrow K_0(A); i \in \mathbb{N}\}$ and unitaries $\{u_i; i \in \mathbb{N}\}$ such that for any n , if denoted by $w_n = u_1 \cdots u_{n-1}$ (assume $u_{-1} = u_0 = 1$), one has

1. For any $x \in \{g_1, \dots, g_i\} \cup \mathcal{G}_1^n$,

$$|\rho_A \circ h_n(x) - \psi(x)| < \frac{\delta_n \delta}{2^n}$$

where

$$\delta_n = \Delta_B\left(\frac{\delta^p}{r_n^2 \cdot 2^{n+1}}, \mathcal{F}_n, \mathcal{P}_n^{(0)}, \mathcal{P}_n^{(1)}, \text{Ad}(w_{n-1}) \circ \iota\right)$$

for $\mathcal{P}_n^{(0)} = \{k_1, k_2, \dots, k_n\}$ and $\mathcal{P}_n^{(1)} = \{g_1, g_2, \dots, g_n\}$, δ the constant of Lemma 3.9 (since A has Property (B1)), and \mathcal{G}_1^n is the set of generators of G_1 of Definition 3.6 with respect to $\frac{\delta^p}{r_n^2 \cdot 2^{n+1}}, \mathcal{F}_n, \mathcal{P}_n^{(0)}, \mathcal{P}_n^{(1)}$, and ι).

2. For any $a \in w_{n-1}^* \mathcal{F}_n w_{n-1}$,

$$\|[u_n, a]\| \leq \frac{\delta^p}{r_n^2 \cdot 2^{n+1}}$$

where $w_n = u_1 \dots u_n$, and moreover, the image of $\phi_n := h_{n+1}|_{H_n} - h_n$ lies inside $C(A)$ so that $\Lambda \circ \phi_n$ is well-defined, and

$$\text{bott}_1((w_{n-1}^{(r_n)})^* z_i w_{n-1}^{(r_n)}, u_n) = \Lambda \circ \phi_n(g_i) \quad \text{and} \quad \text{bott}_0((w_{n-1}^{(r_n)})^* p_i w_{n-1}^{(r_n)}, u_n) = 0$$

for any $i = 1, \dots, n$.

If $n = 1$, since $\rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$, by Lemma 4.1, there is a map $h_1 : \langle \{g_1\} \cup \mathcal{G}_1 \rangle \rightarrow K_0(A)$ such that for any $x \in \{g_1\} \cup \mathcal{G}_1$

$$|\psi(x) - \rho_A(h_1(x))| < \frac{\delta_1 \delta}{2},$$

and a map $h_2 : \langle \{g_1, g_2\} \cup \mathcal{G}_2 \rangle \rightarrow K_0(A)$ such that

$$|\psi(x) - \rho_A(h_2(x))| \leq \frac{\delta_1 \delta}{2^2}$$

for any $x \in \{g_1, g_2\} \cup \mathcal{G}_2$. For the function

$$\phi_1 = h_2|_{H_1} - h_1,$$

we have that $|\tau(\phi_1(g_1))| < \min\{\delta, \delta_1\}$ for any $\tau \in T(A)$ and hence $\phi_1(g_1) \in C(A)$ by Lemma 3.9. By Lemma 3.5,

$$|\tau(\Lambda \circ \phi_1(x))| = |\tau(\phi_1(x))| < \delta_1$$

for any $\tau \in T(A)$ and any $x \in \mathcal{G}_1$. Since A has Property (B2) associated with B and Δ_B , there is a unitary $u_1 \in U(A)$ such that

$$\|[u_1, a]\| \leq \frac{\delta^p}{r_1^2 \cdot 2^2}, \quad \forall a \in \mathcal{F}_1,$$

and

$$\text{bott}_1(z_1, u_1) = \Lambda \circ \phi_1(g_1) \quad \text{and} \quad \text{bott}_0(p_1, u_1) = 0.$$

Assume that we have constructed the maps $\{h_i : H_i \rightarrow K_0(A); i = 1, \dots, n\}$ and unitaries $\{u_i; i = 1, \dots, n-1\}$ satisfying the assertion above. By Lemma 4.1, there is a function $h_{n+1} : \langle \{g_1, \dots, g_{n+1}\} \cup \mathcal{G}_1^n \cup \mathcal{G}_1^{n+1} \rangle \rightarrow K_0(A)$ such that for any $x \in \{g_1, \dots, g_{n+1}\} \cup \mathcal{G}_1^n \cup \mathcal{G}_1^{n+1}$,

$$|\rho_A \circ h_{n+1}(x) - \psi(x)| < \frac{\delta_{n+1} \delta}{2^{n+1}}$$

where

$$\delta_{n+1} = \Delta_B\left(\frac{\delta^p}{r_n^2 \cdot 2^{n+2}}, \mathcal{F}_{n+1}, \mathcal{P}_{n+1}^{(0)}, \mathcal{P}_{n+1}^{(1)}, \text{Ad}(w_n) \circ \iota\right)$$

for $\mathcal{P}_{n+1}^{(0)} = \{k_1, k_2, \dots, k_{n+1}\}$ and $\mathcal{P}_{n+1}^{(1)} = \{g_1, g_2, \dots, g_{n+1}\}$.

Recall that $\phi_n = h_{n+1}|_{H_n} - h_n$. Then, a direct calculation shows that for any $\tau \in T(A)$,

$$|\tau(\Lambda \circ \phi_n)(x)| = |\tau(\phi_n(x))| < \min\{\delta, \delta_{n+1}\}$$

for any $x \in \mathcal{G}_1^n$. Since A Property (B2) associated with B and Δ_B , there is a unitary $u_n \in U(A)$ such that

$$\|[u_n, w_{n-1}^* a w_{n-1}]\| \leq \frac{\delta^p}{r_n^2 \cdot 2^{n+1}}, \quad \forall a \in \mathcal{F}_n,$$

and

$$\text{bott}_1(\text{Ad}(w_{n-1}) \circ \iota, u_n)|_{\mathcal{P}_n^{(1)}} = \Lambda \circ \phi_n \quad \text{and} \quad \text{bott}_0(\text{Ad}(w_{n-1}) \circ \iota, u_n)|_{\mathcal{P}_n^{(0)}} = 0.$$

This proves the assertion.

By Lemma 3.13, $\text{Ad}(w_n)$ converge to a monomorphism $\alpha : B \rightarrow A$. Moreover, the extension $\eta_0(M_\alpha)$ is trivial, and $\eta_1(M_\alpha)$ is determined by the inductive limit of

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} \longrightarrow 0 \\ & & \parallel & & \parallel & \nearrow \gamma_n & \uparrow \iota_{n,n+1} \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n \longrightarrow 0 \end{array}$$

where $\gamma_n(g_i) = [\mathbf{p}(R^*(u_{n+1}, t), w_n^* z_i w_n)]$. However, since

$$\Lambda \circ \gamma_n(g_i) = \text{bott}_1(\text{Ad}(w_{n-1}) \circ \iota, u_n)(g_i) = \Lambda \circ \phi_n(g_i),$$

by the injectivity of Γ , we have that $\gamma_n = \phi_n$.

We assert that $\eta_1(M_\alpha)$ is also trivial. For any n , define a map $\theta'_n : H_n \rightarrow K_1(M_\alpha)$ by

$$\theta'_n(g) = (h_n(g), g).$$

We then have

$$\begin{aligned} & \theta'_{n+1} \circ \iota_{n,n+1}(g) - \theta'_n(g) \\ &= (h_{n+1} \circ \iota_{n,n+1}(g), \iota_{n,n+1}(g)) - (h_n(g) + \phi_n(g), \iota_{n,n+1}(g)) \\ &= (h_{n+1} \circ \iota_{n,n+1}(g), \iota_{n,n+1}(g)) - (h_n(g) + (h_{n+1}(\iota_{n,n+1}(g)) - h_n(g)), \iota_{n,n+1}(g)) \\ &= 0, \end{aligned}$$

and hence (θ'_n) define a homomorphism $\theta_1 : K_1(B) \rightarrow K_1(M_\alpha)$. Moreover, since $\pi \circ \theta_1 = \text{id}_{K_1(B)}$, the extension $\eta_1(M_\alpha)$ splits. Therefore, $[\alpha] = [\iota]$ in $KK(B, A)$.

Let us calculate the corresponding rotation map. Pick $[z_i] = g_i$ for some $g_i \in H_m$. To simplify notation, without loss of generality, we will use α for $\alpha \otimes \text{id}_{M_r(\mathbb{C})}$ on $B \otimes M_r(\mathbb{C})$ for any integer $r \geq 1$. Take a path $v(t)$ in $M_{2r_i}(A)$ from z_i to $\alpha(z_i)$ as follows

$$v(t) = R^*(2t) \begin{pmatrix} 1 & 0 \\ 0 & (w_m^{(r(i))})^* \end{pmatrix} R(2t) \begin{pmatrix} z_i & 0 \\ 0 & 1 \end{pmatrix} R^*(2t) \begin{pmatrix} 1 & 0 \\ 0 & w_m^{(r(i))} \end{pmatrix} R(2t)$$

for $t \in [0, 1/2]$ and

$$v(t) = (w_m^{(r(i))})^* z_i w_m^{(r(i))} \exp((2t - 1)c_m)$$

for $t \in [1/2, 1]$, where

$$R(t) = \begin{pmatrix} \cos(\frac{\pi t}{2}) & \sin(\frac{\pi t}{2}) \\ -\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix}$$

and $c_m = \log((w_m^{(r(i))})^* z_i^* w_m^{(r(i))} \alpha(z_i))$.

Then, for any $\tau \in T(A)$, we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_0^1 \tau(\dot{v}(t)v^*(t))dt &= \tau(\log((w_m^{(r(i))})^* z_i^* w_m^{(r(i))} \alpha(z_i))) \\
&= \lim_{n \rightarrow \infty} \tau(\log((w_m^{(r(i))})^* z_i^* w_m^{(r(i))} \text{Ad} w_n^{(r(i))}(z_i))) \\
&= \lim_{n \rightarrow \infty} \sum_{k=0}^n \tau(\log(((w_{m+k-1}^{(r(i))})^* z_i^* w_{m+k-1}^{(r(i))})((w_{m+k}^{(r(i))})^* z_i w_{m+k}^{(r(i))}))) \\
&= \sum_{k=0}^{\infty} \tau(\log(((w_{m+k-1}^{(r(i))})^* z_i^* w_{m+k-1}^{(r(i))})((w_{m+k}^{(r(i))})^* z_i w_{m+k}^{(r(i))}))) \\
&= \sum_{k=0}^{\infty} \tau(\log(((w_{m+k-1}^{(r(i))})^* z_i^* w_{m+k-1}^{(r(i))})(u_{m+k}^{(r(i))}((w_{m+k-1}^{(r(i))})^* z_i w_{m+k-1}^{(r(i))} u_{m+k}^{(r(i))}))) \\
&= \sum_{k=0}^{\infty} \tau(\text{bott}_1((w_{m+k-1}^{(r(i))})^* z_i^* w_{m+k-1}^{(r(i))}, u_{m+k})).
\end{aligned}$$

Then, by the choice of $\{u_n\}$, we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_0^1 \tau(\dot{v}(t)v^*(t))dt &= \sum_{k=0}^{\infty} \tau(\Lambda \circ \phi_{m+k}(g_i)) \\
&= \sum_{k=0}^{\infty} \tau(\phi_{m+k}(g_i)) \\
&= \sum_{k=0}^{\infty} \tau(h_{m+k+1}(g_i) - h_{m+k}(g_i)) \\
&= \psi(g_i) - \tau(h_m(g_i)).
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{\alpha_1}([v]) &= \psi(g_i) - \rho_A \circ h_m(g_i) \\
&= \psi([\pi_0(v)]) + \rho_A(-h_m(g_i)) \\
&= \psi([\pi_0(v)]) + \rho_A((0, g_i) - \theta'_m(g_i)) \\
&= \psi([\pi_0(v)]) + \rho_A([v] - \theta_1([\pi_0(v)])),
\end{aligned}$$

as desired. \square

Corollary 4.3. *Let A be a unital simple C^* -algebra with $\text{TR}(A) = 0$ and let $B \subseteq A$ be an AH -algebra and denote by ι the inclusion map. For any $\psi \in \text{Hom}(K_1(B), \text{Aff}(T(A)))$, there exists $\alpha \in \overline{\text{Inn}}(B, A)$ such that there are maps $\theta_i : K_i(B) \rightarrow K_i(M_\alpha)$ with $\pi_0 \circ \theta_i = \text{id}_{K_i(B)}$, $i = 0, 1$, and the rotation map $R_{\iota, \alpha} : K_1(M_\alpha) \rightarrow \text{Aff}(T(A))$ is given by*

$$R_{\iota, \alpha}(c) = \rho_A(c - \theta_1([\pi_0]_1(c))) + \psi([\pi_0]_1(c)), \quad \forall c \in K_1(M_\alpha(A)).$$

In other words,

$$[\alpha] = [\iota]$$

in $KK(B, A)$ and the rotation map $R_{\iota, \alpha} : K_1(M_\alpha) \rightarrow \text{Aff}(T(A))$ is given by

$$R_{\iota, \alpha}(a, b) = \rho_A(a) + \psi(b)$$

for some identification of $K_1(M_\alpha)$ with $K_0(A) \oplus K_1(B)$.

Proof. It follows directly from Lemma 5.2 of [15], Theorem 4.2, and Proposition 3.18. \square

Definition 4.4. Fix two unital C^* -algebras A and B with $T(A) \neq \emptyset$. Let \mathcal{R}_0 be the subset of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$ consisting of those homomorphisms $h \in \text{Hom}(K_1(B), \text{Aff}(T(A)))$ such that there exists a homomorphism $d : K_1(B) \rightarrow K_0(A)$ such that

$$h = \rho_A \circ d.$$

Then \mathcal{R}_0 is clearly a subgroup of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$.

The following is a variation of Theorem 10.7 of [15].

Theorem 4.5. Let C be a unital AH-algebra and let A be a unital separable simple C^* -algebra with $\text{TR}(A) = 0$. Suppose that $\phi, \psi : C \rightarrow A$ are two unital monomorphisms such that

$$[\phi] = [\psi]$$

in $KK(C, A)$ and, for all $\tau \in T(A)$,

$$\tau \circ \phi = \tau \circ \psi.$$

Suppose also that there exists a homomorphism $\theta \in \text{Hom}(K_1(C), K_1(M_{\phi, \psi}))$ with $(\pi_0)_* \circ \theta = \text{id}_{K_1(C)}$ such that

$$R_{\phi, \psi} \circ \theta \in \mathcal{R}_0. \quad (4.1)$$

Then ϕ and ψ are asymptotically unitarily equivalent.

Proof. We may write

$$K_1(M_{\phi, \psi}) = K_0(A) \oplus \theta(K_1(C)).$$

Let $h = R_{\phi, \psi} \circ \theta$. If there is a homomorphism $d : K_1(C) \rightarrow K_0(A)$ such that $h = \rho_A \circ d$, define

$$\theta' = \theta - d.$$

Note that θ is a homomorphism, and $(\pi_0)_* \circ \theta = \text{id}_{K_1(C)}$. Then

$$R_{\phi, \psi} \circ \theta' = R_{\phi, \psi} \circ \theta - \rho_A \circ d = 0.$$

Since $[\phi] = [\psi]$ in $KK(C, A)$, there exists an element $\theta_0 \in \text{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\phi, \psi}))$ such that $[\pi_0] \circ [\theta_0] = [\text{id}_{\underline{K}(C)}]$. Define $\theta'_0 : K_i(C) \rightarrow K_i(M_{\phi, \psi})$ by $\theta'_0|_{K_1(C)} = \theta'$ and $\theta'_0|_{K_0(C)} = \theta_0|_{K_0(C)}$. By the UCT, there exists $\theta''_0 \in KL(C, M_{\phi, \psi})$ such that $\Gamma(\theta''_0) = \theta'_0$, where Γ is the map from $KL(C, M_{\phi, \psi})$ onto $\text{Hom}(K_*(C), K_*(M_{\phi, \psi}))$. We will use the identification

$$KL(SC, M_{\phi, \psi}) = \text{Hom}_\Lambda(\underline{K}(SC), \underline{K}(M_{\phi, \psi})).$$

(see [1]). Put

$$x_0 = [\pi_0] \circ \theta''_0 - [\text{id}_{\underline{K}(C)}].$$

Then $\Gamma(x_0) = 0$. Define $\theta_1 = \theta_0'' - \theta_0 \circ x_0 \in \text{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\phi, \psi}))$. Then one computes that

$$[\pi_0] \circ \theta_1 = [\pi_0] \circ \theta_0'' - [\pi_0] \circ \theta_0 \circ x_0 \quad (4.2)$$

$$= ([\text{id}_{\underline{K}(C)}] + x_0) - [\text{id}_{\underline{K}(C)}] \circ x_0 \quad (4.3)$$

$$= [\text{id}_{\underline{K}(C)}] + x_0 - x_0 = [\text{id}_{\underline{K}(C)}]. \quad (4.4)$$

Moreover,

$$\Gamma(\theta_1)|_{K_1(C)} = \theta'.$$

In particular,

$$R_{\phi, \psi} \circ (\theta_1)_{K_1(C)} = 0.$$

It follows from Theorem 10.7 of [15] that ϕ and ψ are asymptotically unitarily equivalent. \square

Remark 4.6. In 4.5, the condition that such θ exists is also necessary. This follows from Theorem 9.1 of [15].

Definition 4.7. Let A be a unital C^* -algebra, and let C be a unital separable C^* -algebra. Denote by $\text{Mon}_{asu}^e(C, A)$ the set of asymptotically unitary equivalence classes of unital monomorphisms.

Denote by \mathbf{K} the map from $\text{Mon}_{asu}^e(C, A)$ into $KK_e(C, A)^{++}$ defined by

$$\phi \mapsto [\phi] \text{ for all } \phi \in \text{Mon}_{asu}^e(C, A).$$

Let $\kappa \in KK_e(C, A)^{++}$. Denote by $\langle \kappa \rangle$ the classes of $\phi \in \text{Mon}_{asu}^e(C, A)$ such that $\mathbf{K}(\phi) = \kappa$.

Denote by $\widetilde{\mathbf{K}}$ the map from $\text{Mon}_{asu}^e(C, A)$ into $KKT(C, A)^{++}$ defined by

$$\phi \mapsto ([\phi], \phi_T) \text{ for all } \phi \in \text{Mon}_{asu}^e(C, A).$$

Denote by $\langle \kappa, \lambda \rangle$ the classes of $\phi \in \text{Mon}_{asu}^e(C, A)$ such that $\widetilde{\mathbf{K}}(\phi) = (\kappa, \lambda)$.

Theorem 4.8. Let C be a unital AH -algebra and let A be a unital separable simple C^* -algebra with $\text{TR}(A) = 0$. Then the map $\widetilde{\mathbf{K}} : \text{Mon}_{asu}^e(C, A) \rightarrow KKT(C, A)^{++}$ is surjective. Moreover, for each $(\kappa, \lambda) \in KKT(C, A)^{++}$, there exists a bijection

$$\eta : \langle \kappa, \lambda \rangle \rightarrow \text{Hom}(K_1(C), \text{Aff}(T(A))) / \mathcal{R}_0.$$

Proof. It follows from 3.22 that $\widetilde{\mathbf{K}}$ is surjective.

Fix a pair $(\kappa, \lambda) \in KKT(C, A)^{++}$ and fix a unital monomorphism $\phi : C \rightarrow A$ such that $[\phi] = \kappa$ and $\phi_T = \lambda$. For any homomorphism $\gamma \in \text{Hom}(K_1(C), \text{Aff}(T(A)))$, it follows from 4.2 that there exists a unital monomorphism $\alpha \in \overline{\text{Inn}}(\phi(C), A)$ with $[\alpha \circ \phi] = [\phi]$ in $KK(C, A)$ such that there exists a homomorphism $\theta : K_1(C) \rightarrow M_{\phi, \alpha \circ \phi}$ with $(\pi_0)_* \circ \theta = \text{id}_{K_1(C)}$ such that $R_{\phi, \alpha \circ \phi} \circ \theta = \gamma$. Let $\psi = \alpha \circ \phi$. Then $R_{\phi, \psi} \circ \theta = \gamma$. Note also since $\alpha \in \overline{\text{Inn}}(\phi(C), A)$, $\psi_T = \phi_T$. In particular, $\widetilde{\mathbf{K}}(\psi) = \widetilde{\mathbf{K}}(\phi)$.

Suppose that $\theta' : K_1(A) \rightarrow K_1(M_{\phi, \psi})$ such that $(\pi_0)_* \circ \theta' = \text{id}_{K_1(C)}$. Then

$$(\theta' - \theta)(K_1(C)) \subset K_0(A).$$

It follows that $R_{\phi, \psi} \circ \theta' - R_{\phi, \psi} \circ \theta \in \mathcal{R}_0$.

Thus we obtain a well-defined map $\eta : \langle [\phi], \phi_T \rangle \rightarrow \text{Hom}(K_1(A), \text{Aff}(\text{T}(A))) / \mathcal{R}_0$. From what we have proved, the map η is surjective.

Suppose that $\phi_1, \phi_2 : C \rightarrow A$ are two unital monomorphisms such that $\phi_1, \phi_2 \in \langle [\phi], \phi_T \rangle$ and

$$R_{\phi, \phi_1} \circ \theta_1 - R_{\phi, \phi_2} \circ \theta_2 \in \mathcal{R}_0,$$

where $\theta_1 : K_1(C) \rightarrow K_1(M_{\phi, \phi_1})$ and $\theta_2 : K_1(C) \rightarrow K_1(M_{\phi, \phi_2})$ are homomorphisms such that $(\pi_0)_{*1} \circ \theta_1 = (\pi_0)_{*1} \circ \theta_2 = \text{id}_{K_1(C)}$, respectively. Thus there is $\delta : K_1(C) \rightarrow K_0(A)$ such that

$$R_{\phi, \phi_2} \circ \theta_2 - R_{\phi, \phi_1} \circ \theta_1 = \rho_A \circ \delta.$$

Fix $z \in K_1(C)$ and let $u \in M_k(C)$ such that $[u] = z$ in $K_1(C)$. Let $U(\theta_1, z)(t) \in \text{U}(M_N(M_{\phi, \phi_1}))$ be a unitary represented by $\theta_1(z)$. We may assume that $N \geq k$. It is easy to see that $U(\theta_1, z)(t)$ is homotopic to a unitary $V(\theta_1, z)(t) \in \text{U}(M_N(M_{\phi, \phi_1}))$ with $V(\theta_1, z)(0) = \text{diag}(\phi(u), 1_{N-k})$ and $V(\theta_1, z)(1) = \text{diag}(\phi_1(u), 1_{N-k})$ (by enlarging the size of matrices if necessary). In particular, $[V(\theta_1, z)] = \theta_1(z)$ in $K_1(M_{\phi, \phi_1})$. Similarly, we may assume that there is a unitary $V(\theta_2, z) \in \text{U}(M_N(M_{\phi, \phi_2}))$ with $V(\theta_2, z)(0) = \text{diag}(\phi(u), 1_{N-k})$ and $V(\theta_2, z)(1) = \text{diag}(\phi_2(u), 1_{N-k})$ which is represented by $\theta_2(z)$ in $K_1(M_{\phi, \phi_2})$.

Now define

$$V(\theta_3, z)(t) = \begin{cases} V(\theta_1, z)(1-2t) & \text{for } t \in [0, 1/2] \\ V(\theta_2, z)(2t-1) & \text{for } t \in (1/2, 1], \end{cases} \quad (4.5)$$

Note that $V(\theta_3, z) \in \text{U}(M_N(M_{\phi_1, \phi_2}))$.

Define the homomorphism $\theta_3 : K_1(C) \rightarrow K_1(M_{\phi_1, \phi_2})$ by

$$\theta_3(z) = [V(\theta_3, z)] \text{ for all } z \in K_1(C).$$

It follows that

$$(\pi_0)_{*1} \circ \theta_3 = (\pi_0)_{*1} \circ \theta_1 = \text{id}_{K_1(C)}.$$

Moreover, a direct calculation shows that

$$R_{\phi_1, \phi_2} \circ \theta_3 = R_{\phi, \phi_2} \circ \theta_2 - R_{\phi, \phi_1} \circ \theta_1 = \rho_A \circ \delta.$$

It follows from Theorem 4.5 that ψ_1 and ψ_2 are asymptotically unitarily equivalent. Therefore, η is one to one. \square

Corollary 4.9. *Let C be a unital separable amenable simple C^* -algebra with $\text{TR}(C) = 0$ which satisfies the UCT and let A be a unital simple C^* -algebra with $\text{TR}(A) = 0$. Then the map $\mathbf{K} : \text{Mon}_{asu}^e(C, A) \rightarrow KK_e(C, A)^{++}$ is surjective. Moreover, the map*

$$\eta : \langle [\phi] \rangle \rightarrow \text{Hom}(K_1(C), \text{Aff}(\text{T}(A))) / \mathcal{R}_0$$

is bijective.

Proof. It follows from 4.8 that it suffices to point out that in this case $\rho_C(K_0(C))$ is dense in $\text{Aff}(\text{T}(C))$ and $\text{T}_{\mathbf{f}}(C) = \text{T}(C)$. \square

Corollary 4.10. *Let A be a unital separable amenable simple C^* -algebra with $\text{TR}(A) = 0$ which satisfies the UCT. Then the map $\mathbf{K} : \text{Aut}(A) \rightarrow KK_e^{-1}(A, A)^{++}$ is surjective. Moreover, the map*

$$\eta : \langle \text{id}_A \rangle \rightarrow \text{Hom}(K_1(A), \text{Aff}(T(A))) / \mathcal{R}_0$$

is bijective.

Proof. This follows from 3.26 and 4.2 as in the proof of 4.8. \square

5 Classification of simple C^* -algebras

Denote by Q the UHF algebra with $K_0(Q) = \mathbb{Q}$ with $[1_Q] = 1$. If \mathfrak{p} is a supernatural number, let $M_{\mathfrak{p}}$ denote the UHF-algebra associated with \mathfrak{p} .

Lemma 5.1. *Let B be a unital separable amenable simple C^* -algebra such that $B \otimes Q$ has tracial rank zero. Let A be unital separable amenable simple C^* -algebras with tracial rank zero satisfying the UCT, and let $\varphi_1, \varphi_2 : A \rightarrow B$ be two homomorphisms. Suppose that $\phi, \psi : A \rightarrow B \otimes Q$ are two unital monomorphisms such that*

$$[\phi] = [\psi] \text{ in } KK(A, B \otimes Q).$$

Suppose that ϕ induces an affine homeomorphism $\phi_T : T(B \otimes Q) \rightarrow T(A)$ by

$$\phi_T(\tau)(a) = \tau \circ \phi(a) \text{ for all } a \in A_{s,a}$$

and for all $\tau \in T(B \otimes Q)$. Then there exists an automorphism $\alpha \in \text{Aut}(\phi(A), \phi(A))$ with $[\alpha] = [\text{id}_{\phi(A)}]$ in $KK(\phi(A), \phi(A))$ such that $\alpha \circ \phi$ and ψ are strongly asymptotically unitarily equivalent.

Proof. The proof is exactly the same as that of the proof of Lemma 3.2 of [16] but we apply 4.2 instead of 4.1 of [8] so that the restriction on $K_i(A)$ ($i = 0, 1$) can be removed. \square

Definition 5.2. Let C and A be two unital C^* -algebras. Let $\phi, \psi : C \rightarrow A$ be two homomorphisms. Recall that ϕ and ψ are said to be strongly asymptotically unitarily equivalent if there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\}$ such that $u(0) = 1$ and

$$\lim_{t \rightarrow \infty} \text{ad } u(t) \circ \phi(c) = \psi(c) \text{ for all } c \in C.$$

Lemma 5.3 (Theorem 3.4 of [16]). *Let A and B be two unital separable amenable simple C^* -algebras satisfying the UCT. Let \mathfrak{p} and \mathfrak{q} be supernatural numbers of infinite type such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$. Suppose that $A \otimes M_{\mathfrak{p}}, A \otimes M_{\mathfrak{q}}, B \otimes M_{\mathfrak{p}}$ and $B \otimes M_{\mathfrak{q}}$ have tracial rank zero.*

Let $\sigma_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \rightarrow B \otimes M_{\mathfrak{p}}$ and $\rho_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \rightarrow B \otimes M_{\mathfrak{q}}$ be two unital isomorphisms. Suppose

$$[\sigma] = [\rho]$$

in $KK(A \otimes Q, B \otimes Q)$, where $\sigma = \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$ and $\rho = \rho_{\mathfrak{q}} \otimes \text{id}_{M_{\mathfrak{p}}}$.

Then there is an automorphism $\alpha \in \text{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ such that

$$[\alpha \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}]$$

in $KK(A \otimes M_{\mathfrak{p}}, B \otimes M_{\mathfrak{p}})$, and $\alpha \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$ is strongly asymptotically unitarily equivalent to ρ .

Proof. It follows from 5.1 that there exists $\beta \in \text{Aut}(B \otimes Q)$ such that $\beta \circ \sigma$ is strongly asymptotically unitarily equivalent to ρ . Moreover, $[\beta] = [\text{id}_{B \otimes Q}]$ in $KK(B \otimes Q, B \otimes Q)$. Now consider two homomorphisms $\sigma_{\mathfrak{p}}$ and $\beta \circ \sigma_{\mathfrak{p}}$. One has

$$[\beta \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}] \text{ in } KK(A \otimes M_{\mathfrak{p}}, B \otimes Q).$$

Since $\sigma_{\mathfrak{p}}$ is an isomorphism, it is easy to see that $\sigma_T : T(B \otimes Q) \rightarrow T(A \otimes M_{\mathfrak{p}})$ is an affine homeomorphism.

By applying 5.1 again, one obtains $\alpha \in \text{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ such that $[\alpha] = [\text{id}_{\sigma_{\mathfrak{p}}}]$ in $KK(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}), \sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ and $\alpha \circ \sigma_{\mathfrak{p}}$ is strongly asymptotically unitarily equivalent to $\beta \circ \sigma_{\mathfrak{p}}$.

Define $\beta \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}} : A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \rightarrow (B \otimes Q) \otimes M_{\mathfrak{q}}$. It is easy to see that $\beta \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$ is strongly approximately unitarily equivalent to $\beta \circ \sigma$.

Note that $\sigma(A \otimes M_{\mathfrak{p}}) = B \otimes M_{\mathfrak{p}}$. Let $\sigma' = \alpha \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$. It follows that σ' is strongly asymptotically unitarily equivalent to $\beta \circ \sigma$. Consequently σ' is strongly asymptotically unitarily equivalent to ρ . \square

Theorem 5.4. *Let A and B be two unital separable amenable simple C^* -algebras satisfying the UCT. Suppose that there is an isomorphism*

$$\kappa : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Suppose also that there is a pair of super-natural numbers \mathfrak{p} and \mathfrak{q} of infinite type which are relative prime such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$ and

$$\text{TR}(A \otimes M_{\mathfrak{p}}) = \text{TR}(A \otimes M_{\mathfrak{q}}) = \text{TR}(B \otimes M_{\mathfrak{p}}) = \text{TR}(B \otimes M_{\mathfrak{q}}) = 0.$$

Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.

Proof. The proof is exactly the same as that of Theorem 3.5 of [16] but we now apply 5.3 instead. \square

Corollary 5.5 (Corollary 8.3 of [24]). *Let A and B be two unital separable simple ASH-algebras whose projections separate traces which are \mathcal{Z} -stable. Suppose that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Then $A \cong B$.

Proof. As in the proof of 6.3 of [24], $A \otimes C$ and $B \otimes C$ have tracial rank zero for any unital UHF-algebra C . Thus Theorem 5.4 applies. \square

Corollary 5.6. *Let A and B be two unital separable simple \mathcal{Z} -stable C^* -algebras which are inductive limits of type I C^* -algebras with unique tracial states. Suppose that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Then $A \cong B$.

Proof. For any UHF-algebra C , $A \otimes C$ is approximately divisible. Since A has a unique tracial state, so does $A \otimes C$. Therefore projections of $A \otimes C$ separate traces. It follows from [21] that $A \otimes C$ has real rank zero, stable rank one and weakly unperforated $K_0(A \otimes C)$. Moreover, $A \otimes C$ is also an inductive limit of type I C^* -algebras. It follows from Theorem 4.15 and Proposition 5.4 of [13] that $\text{TR}(A \otimes C) = 0$. Exactly the same argument shows that $\text{TR}(B \otimes C) = 0$. It follows from Theorem 5.4 that $A \cong B$. \square

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